Definition:

- **Logarithmic function**: Let a be a positive number with $a \neq 1$. The logarithmic function with base a, denoted $\log_a x$, is defined by

$$\boxed{y = \log_a x \qquad \text{if and only if} \qquad x = a^y.}$$

Important Formulas:

- **Compound Interest**: is calculated by the formula

$$\boxed{A(t) = P\left(1 + \frac{r}{n}\right)^{nt}}$$

where

$$\begin{aligned} A(t) &= \text{ amount after } t \text{ years} \\ P &= \text{ principal} \\ r &= \text{ interest rate} \\ n &= \text{ number of times the interest is compounded per year} \\ t &= \text{ number of years} \end{aligned}$$

- **Compounded Continuously Interest**: is calculated by the formula

$$\boxed{A(t) = Pe^{rt}}$$

where

$$\begin{aligned} A(t) &= \text{ amount after } t \text{ years} \\ P &= \text{ principal} \\ r &= \text{ interest rate} \\ t &= \text{ number of years} \end{aligned}$$

- **Exponential Growth**: of a population increases according to the formula

$$\boxed{P(t) = P_0 e^{rt}}$$

where

$$\begin{aligned} P(t) &= \text{ population after time } t \\ P_0 &= \text{ initial population} \\ r &= \text{ growth rate} \\ t &= \text{ time} \end{aligned}$$

Exponential Decay: of a substance is given by the following formula

$$\boxed{m(t) = m_0 e^{-rt}}$$

where

$$\begin{aligned} m(t) &= \text{ mass remaining after time } t \\ m_0 &= \text{ initial mass} \\ r &= \text{ decay rate} \\ t &= \text{time} \end{aligned}$$

Its half-life is given by $h = \frac{\ln 2}{r}$.

PROBLEMS

1. **How long will it take for an investment of $\$2000$ to double in value if the interest rate is 7.25% per year, compounded continuously?**

   Here, $P = 2000$ so $A(t) = 4000$. Also, $r = .0725$ and t is what we are solving for. Substituting all known values into the compounded continuous interest formula, we get

$$\begin{aligned} A(t) &= Pe^{rt} \\ 4000 &= 2000e^{.0725t} \\ 2 &= e^{.0725t} \\ \ln 2 &= \ln e^{.0725t} \\ \ln 2 &= .0725t \ln e \\ \ln 2 &= .0725t \\ \frac{\ln 2}{.0725} &= \frac{.0725t}{.0725} \\ \frac{\ln 2}{.0725} &= t \\ 9.560650766 &= t \end{aligned}$$

$$\boxed{\text{Approximately 9.5 years}}$$

2. **The deer population at the local reserve grows exponentially. The current population is 125 deer and the relative growth rate is 16% per year. Find the number of years required for the deer population to be 400.**

   Here, $P_0 = 125, \quad r = .16$ and we want to find t so that $A(t) = 400$. Substituting into the exponential growth formula and solving for t, we get

$$\begin{aligned} P(t) &= P_0 e^{rt} \\ 400 &= 125e^{.16t} \\ 3.2 &= e^{.16t} \\ \ln 3.2 &= \ln e^{.16t} \\ \ln 3.2 &= .16t \ln e \\ \ln 3.2 &= .16t \\ \frac{\ln 3.2}{.16} &= t \\ 7.269692561 &= t \end{aligned}$$

$$\boxed{\text{Approximately 7.3 years}}$$

3. **Oskie-946 has a decay rate of 13.5%. If the original sample was 50 grams, how long will it take for only 10 grams of the sample to remain?**

Here, we know that $r = .135$, $m_0 = 50$ and we want to find t so that $m(t) = 10$. Substituting into the exponential decay formula, we get

$$\begin{aligned} m(t) &= m_0 e^{-rt} \\ 10 &= 50e^{-.135t} \\ .2 &= e^{-.135t} \\ \ln .2 &= \ln e^{-.135t} \\ \ln .2 &= -.135t \ln e \\ \ln .2 &= -.135t \\ \frac{\ln .2}{-.135} &= t \\ 11.92176231 &= t \end{aligned}$$

$$\boxed{\text{Approximately 12 years}}$$

4. **If a 325 mg sample of radioactive material decays to 195 mg in 72 hours, find the half-life of the element.**

Recall, that half-life $h = \frac{\ln 2}{r}$. Therefore, we need to find the decay rate r. To do this, we substitute $m_0 = 325$, $m(t) = 195$ and $t = 72$ into the exponential decay formula and solve for r.

$$\begin{aligned} m(t) &= m_0 e^{-rt} \\ 195 &= 325e^{-72r} \\ .6 &= e^{-72r} \\ \ln .6 &= \ln e^{-72r} \\ \ln .6 &= -72r \ln e \\ \ln .6 &= -72r \\ \frac{\ln .6}{-72} &= r \\ .0070948003 &= r \end{aligned}$$

Therefore, substituting this into the formula for half-life, we get

$$h = \frac{\ln 2}{r} = \frac{\ln 2}{.0070948003} = 97.69791232.$$

$$\boxed{\text{Half-life is approximately 97.7 hours}}$$

5. **If $2500 was invested 6 years ago, and the interest was compounded quarterly, what was the interest rate if the current value is $3425?**

Here, $P_0 = 2500$, $t = 6$, $n = 4$, and $A(t) = 3425$. Substituting into the compound interest formula we get

$$A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

$$3425 = 2500\left(1 + \frac{r}{4}\right)^{4\cdot 6}$$

$$3425 = 2500\left(1 + \frac{r}{4}\right)^{24}$$

$$1.37 = \left(1 + \frac{r}{4}\right)^{24}$$

$$(1.37)^{1/24} = \left(1 + \frac{r}{4}\right)$$

$$1.013203521 = 1 + \frac{r}{4}$$

$$.013203521 = \frac{r}{4}$$

$$4(.013203521) = r$$

$$.0528140836 = r$$

Interest rate approximately 5.28%

6. **If $2500 is invested at an interest rate of 6.5%, compounded monthly, how long will it take for the investment to reach $7500?**

Here, $P = 2500$, $r = .065$, $n = 12$ and we want to find t so that $A(t) = 7500$. Substituting into the compound interest formula, we get

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$7500 = 2500\left(1 + \frac{.065}{12}\right)^{12t}$$

$$3 = (1.005416667)^{12t}$$

$$\log 3 = \log(1.005416667)^{12t}$$

$$\log 3 = 12t\log(1.005416667)$$

$$\frac{\log 3}{12\log(1.005416667)} = t$$

$$16.94746078 = t$$

Approximately 17 years