VERIFYING THE EQUATION OF A CIRCLE BY COMPLETING THE SQUARE

Definitions:

- **Circle**: is the set of all points in a plane that lie a fixed distance from a fixed point. The fixed distance is called the **radius** and the fixed point is called the **center**.

Important Properties:

- **Equation of a circle**: An equation of the circle with center (h, k) and radius r is given by

$$\boxed{(x-h)^2 + (y-k)^2 = r^2.}$$

  This is called the standard form for a circle.

- Note to find the equation of a circle you need two items: the center and the radius.

Steps to verify the equation of a circle by completing the square:

1. Isolate the constant on one side of the equation and group all x terms together and group all y terms together.
2. Take one-half the coefficient of x and square it. Also, take one-half the coefficient of y and square it. Namely,
$$\left(\frac{1}{2} \cdot \text{coeff of } x\right)^2 \qquad \text{and} \qquad \left(\frac{1}{2} \cdot \text{coeff of } y\right)^2$$
3. Add the result of each part of step 2 to both sides of the equation.
4. Factor as two perfect squares.

PROBLEMS

Show that the equation represents a circle, and find the center and radius of the circle.

1. $x^2 + y^2 - 2x + 6y + 3 = 0$

To show that this is the equation of a circle we will need to put it in standard form. To do this we will need to complete the square on both the x and y terms.

$$x^2 + y^2 - 2x + 6y + 3 = 0$$

$$x^2 - 2x + \quad + y^2 + 6y + \quad = -3$$

$$\left(\frac{1}{2} \cdot -2\right)^2 = (-1)^2 = 1$$

$$\left(\frac{1}{2} \cdot 6\right)^2 = (3)^2 = 9$$

$$x^2 - 2x + 1 + y^2 + 6y + 9 = -3 + 1 + 9$$

$$(x-1)^2 + (y+3)^2 = 7$$

This is the standard form for a circle. The center is $(1, -3)$ and the radius is

$$r^2 = 7$$

$$\sqrt{r^2} = \sqrt{7}$$

$$r = \sqrt{7}$$

$$\boxed{(x-1)^2 + (y+3)^2 = 7}$$

$$\boxed{\text{Center} = (1, -3), \qquad r = \sqrt{7}}$$

2. $x^2 + y^2 - 4x + 12y - 7 = 0$

$$x^2 + y^2 - 4x + 12y - 7 = 0$$

$$x^2 - 4x + \quad + y^2 + 12y + \quad = 7$$

$$\left(\frac{1}{2} \cdot -4\right)^2 = (-2)^2 = 4$$

$$\left(\frac{1}{2} \cdot 12\right)^2 = (6)^2 = 36$$

$$x^2 - 4x + 4 + y^2 + 12y + 36 = 7 + 4 + 36$$

$$(x-2)^2 + (y+6)^2 = 47$$

This is the standard form for a circle. The center is $(2, -6)$ and the radius is

$$r^2 = 47$$

$$\sqrt{r^2} = \sqrt{47}$$

$$r = \sqrt{47}$$

$$\boxed{(x-2)^2 + (y+6)^2 = 47}$$

$$\boxed{\text{Center} = (2, -6), \qquad r = \sqrt{47}}$$

3. $x^2 + y^2 + 8x - 2 = 0$

For this problem notice that we only need to complete the square on the x variable since the y^2 is already a perfect square.

$$x^2 + y^2 + 8x - 2 = 0$$
$$x^2 + 8x + \quad +y^2 = 2$$
$$\left(\frac{1}{2} \cdot 8\right)^2 = (4)^2 = 16$$
$$x^2 + 8x + 16 + y^2 = 2 + 16$$
$$(x+4)^2 + y^2 = 18$$

This is the standard form for a circle. The center is $(-4, 0)$ and the radius is

$$r^2 = 18$$
$$\sqrt{r^2} = \sqrt{18}$$
$$r = \sqrt{18}$$
$$r = 3\sqrt{2}$$

$$\boxed{(x+4)^2 + y^2 = 18}$$

$$\boxed{\text{Center} = (-4, 0), \qquad r = 3\sqrt{2}}$$

4. $x^2 + y^2 - 3x - 10y - 13 = 0$

$$x^2 + y^2 - 3x - 10y - 13 = 0$$
$$x^2 - 3x + \quad +y^2 - 10y + \quad = 13$$
$$\left(\frac{1}{2} \cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$
$$\left(\frac{1}{2} \cdot -10\right)^2 = (-5)^2 = 25$$
$$x^2 - 3x + \frac{9}{4} + y^2 - 10y + 25 = 13 + \frac{9}{4} + 25$$
$$\left(x - \frac{3}{2}\right)^2 + (y-5)^2 = \frac{161}{4}$$

This is the standard form for a circle. The center is $\left(\frac{3}{2}, 5\right)$ and the radius is

$$r^2 = \frac{161}{4}$$
$$\sqrt{r^2} = \sqrt{\frac{161}{4}}$$
$$r = \frac{\sqrt{161}}{2}$$

$$\boxed{\left(x - \frac{3}{2}\right)^2 + (y-5)^2 = \frac{161}{4}}$$

$$\boxed{\text{Center} = \left(\frac{3}{2}, 5\right), \qquad r = \frac{\sqrt{161}}{2}}$$

5. $x^2 + y^2 + 9x - 3y + 11 = 0$

$$x^2 + y^2 + 9x - 3y + 11 = 0$$

$$x^2 + 9x + \quad + y^2 - 3y + \quad = -11$$

$$\left(\frac{1}{2} \cdot 9\right)^2 = \left(\frac{9}{2}\right)^2 = \frac{81}{4}$$

$$\left(\frac{1}{2} \cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$x^2 + 9x + \frac{81}{4} + y^2 - 3y + \frac{9}{4} = -11 + \frac{81}{4} + \frac{9}{4}$$

$$\left(x + \frac{9}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{46}{4}$$

This is the standard form for a circle. The center is $\left(-\frac{9}{2}, \frac{3}{2}\right)$ and the radius is found by

$$r^2 = \frac{46}{4}$$

$$\sqrt{r^2} = \sqrt{\frac{46}{4}}$$

$$r = \frac{\sqrt{46}}{2}$$

$$\boxed{\left(x + \frac{9}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{46}{4}}$$

$$\boxed{\text{Center} = \left(-\frac{9}{2}, \frac{3}{2}\right), \qquad r = \frac{\sqrt{46}}{2}}$$