Definition:

- **Logarithmic function**: Let a be a positive number with $a \neq 1$. The logarithmic function with base a, denoted $\log_a x$, is defined by

$$\boxed{y = \log_a x \qquad \text{if and only if} \qquad x = a^y.}$$

Guidelines for solving logarithmic equations:

1. Isolate the logarithmic term on one side of the equation. This is accomplished by using the laws of logarithms.
2. Write the equation in exponential form.
3. Solve for the variable.
4. Check to make sure you don't have extraneous solutions. To do this, substitute "answers" into the original equation and check that you are not taking the logarithm of a negative number or zero.

   (NOTE: In some instances you can solve the logarithmic equation by using the one-to-one property of logarithms.)

Important Properties:

- **One-to-one property of logarithms**: If $a > 0, a \neq 1, x > 0$, and $y > 0$ then

$$\boxed{x = y \quad \text{if and only if} \quad \log_a x = \log_a y.}$$

  We will abbreviate this as the 1-1 prop in our problems.

Common Mistakes to Avoid:

- Be careful not to combine terms that are outside the logarithm with terms that are inside the logarithm. For example,

$$\log 2 + 5 \neq \log 7 \qquad \text{and} \qquad \frac{\log 4}{2} \neq \log 2.$$

- Be careful not to divide out terms that are inside two different logarithms. For example,

$$\frac{\log 15}{\log 3} \neq \log 5 \qquad \text{but} \qquad \log \frac{15}{3} = \log 5.$$

- Do not exclude possible answer just because they are negative numbers. Negative numbers can be solutions to a logarithmic equation as long as when you substitute the value back into the original equation you are not taking the logarithm of a negative number or zero. For example, $x = -2$ is a solution to $\log_2(-x) = 1$.

- You cannot use the one-to-one property on every logarithmic equation.

PROBLEMS

Solve for x in each of the following equations.

1. $\log(3x-2)=2$

$$\log(3x-2)=2$$
$$3x-2=10^2$$
$$3x-2=100$$
$$3x=102$$
$$x=\frac{102}{3}$$

Checking $x=\frac{102}{3}$ in the original equation, we see that it works.

$$\boxed{x=\frac{102}{3}}$$

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2. $\ln(x^2-20)=\ln x$

$$\ln(x^2-20)=\ln x$$
$$\ln(x^2-20)-\ln x=0$$
$$\ln\frac{x^2-20}{x}=0$$
$$\frac{x^2-20}{x}=e^0$$
$$\frac{x^2-20}{x}=1$$
$$x^2-20=x$$
$$x^2-x-20=0$$
$$(x-5)(x+4)=0$$

Setting each factor equal to zero, we get

$x-5=0$	$x+4=0$
$x=5$	$x=-4$

Checking $x=5$ and $x=-4$ back in the original equation, we see that $x=-4$ cannot be a solution since we cannot evaluate $\ln -4$.

$$\boxed{x=5}$$

OR (for an alternative way using one-to-one property of logs)

$$\ln(x^2-20)=\ln x$$
$$x^2-20=x \quad \text{by 1-1 prop}$$
$$x^2-x-20=0$$
$$(x-5)(x+4)=0$$

Setting each factor equal to zero, we get

$x-5=0$	$x+4=0$
$x=5$	$x=-4$

Checking $x=5$ and $x=-4$ back in the original equation, we see that $x=-4$ cannot be a solution since we cannot evaluate $\ln -4$.

$$\boxed{x=5}$$

3. $(\ln x)^2 = \ln x^2$

$$\begin{aligned}(\ln x)^2 &= \ln x^2\\ (\ln x)^2 - \ln x^2 &= 0\\ (\ln x)^2 - 2\ln x &= 0\\ (\ln x)(\ln x - 2) &= 0\end{aligned}$$

Setting each factor equal to zero, we get

$\ln x = 0$ $x = e^0$ $x = 1$	$\ln x - 2 = 0$ $\ln x = 2$ $x = e^2$

Checking both $x = 1$ and $x = e^2$, we see that both are acceptable.

$$\boxed{x = 1, \qquad x = e^2}$$

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4. $\log(x+6) - \log x = \log(x+2)$

To solve this we will clean up each side and use the one-to-one property of logarithms.

$$\begin{aligned}\log(x+6) - \log x &= \log(x+2)\\ \log\frac{x+6}{x} &= \log(x+2)\\ \frac{x+6}{x} &= x+2 \qquad \text{by 1-1 prop}\\ x+6 &= x(x+2)\\ x+6 &= x^2+2x\\ 0 &= x^2+x-6\\ 0 &= (x+3)(x-2)\end{aligned}$$

Setting each factor equal to zero, we get

$x + 3 = 0$ $x = -3$	$x - 2 = 0$ $x = 2$

Checking both answers in the original equation, we see that $x = -3$ cannot work since we cannot evaluate $\log -3$.

$$\boxed{x = 2}$$

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5. $\log x + \log(x-15) = 2$

$$\begin{aligned}\log x + \log(x-15) &= 2\\ \log x(x-15) &= 2\\ x(x-15) &= 10^2\\ x^2 - 15x &= 100\\ x^2 - 15x - 100 &= 0\\ (x-20)(x+5) &= 0\end{aligned}$$

Setting each factor equal to zero, we get

$x - 20 = 0$ $x = 20$	$x + 5 = 0$ $x = -5$

Checking both answers back in the original equation, we observe that $x = -5$ cannot be a solution since we cannot evaluate $\log -5$.

$$\boxed{x = 20}$$

6. $\log_7(2x-1) + \log_7 3 = \log_7(5x+3)$

$$\begin{aligned} \log_7(2x-1) + \log_7 3 &= \log_7(5x+3) \\ \log_7 3(2x-1) &= \log_7(5x+3) \\ 3(2x-1) &= 5x+3 \quad \text{by 1-1 prop} \\ 6x-3 &= 5x+3 \\ x-3 &= 3 \\ x &= 6 \end{aligned}$$

Checking $x = 6$ in the original equation we see that it works.

$$\boxed{x = 6}$$

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7. $\log_2 x - \log_2(x+3) = 1$

$$\begin{aligned} \log_2 x - \log_2(x+3) &= 1 \\ \log_2 \frac{x}{x+3} &= 1 \\ \frac{x}{x+3} &= 2^1 \\ x &= 2(x+3) \\ x &= 2x+6 \\ -x &= 6 \\ x &= -6 \end{aligned}$$

Checking $x = -6$ in the original equation, we see that it will not work since we cannot evaluate $\log_2 -6$.

$$\boxed{\text{No solution}}$$

8. $\log_3(x-4) + \log_3(x+4) = 2$

$$\begin{aligned} \log_3(x-4) + \log_3(x+4) &= 2 \\ \log_3(x-4)(x+4) &= 2 \\ (x-4)(x+4) &= 3^2 \\ x^2 - 16 &= 9 \\ x^2 &= 25 \\ \sqrt{x^2} &= \sqrt{25} \\ x &= \pm 5 \end{aligned}$$

Checking both answers in the original equation, we see that $x = -5$ cannot work because we cannot evaluate $\log_2 -1$.

$$\boxed{x = 5}$$

9. $\ln x + \ln(x+3) = 1$

$$\begin{aligned} \ln x + \ln(x+3) &= 1 \\ \ln x(x+3) &= 1 \\ x(x+3) &= e^1 \\ x^2 + 3x &= e \\ x^2 + 3x - e &= 0 \end{aligned}$$

Since this quadratic does not factor we will solve it using the quadratic formula where $a = 1,\ b = 3$, and $c = -e$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-e)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{9 + 4e}}{2} \end{aligned}$$

Now $x = \dfrac{-3 + \sqrt{9+4e}}{2} \approx .7289$ and $x = \dfrac{-3 - \sqrt{9+4e}}{2} \approx -3.7289$. Checking both answers in the original equation, we find that $x = \dfrac{-3 - \sqrt{9+4e}}{2}$ will not work.

$$\boxed{x = \frac{-3 + \sqrt{9+4e}}{2}}$$