IDENTIFYING THE EXTREME VALUES OF QUADRATIC FUNCTIONS

Definitions:

- **Quadratic function**: is a function that can be written in the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$.

- **Parabola**: The graph of a squaring function is called a parabola. It is a U-shaped graph.

- **Vertex of a parabola**: The point on the parabola where the graph changes direction. It is the lowest point if $a > 0$, and it is the highest point if $a < 0$.

- **Standard form of a quadratic function**: A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the standard form

$$\boxed{f(x) = a(x-h)^2 + k}$$

by completing the square.

Important Properties:

- **Extreme values of a quadratic function**: Consider the quadratic function $f(x) = a(x-h)^2 + k$.
  - If $a > 0$, then the parabola opens up. Therefore, the **minimum value** of f occurs at $x = h$ and its value is $f(h) = k$.
  - If $a < 0$, the parabola opens down. Therefore, the **maximum value** of f occurs at $x = h$ and its value is $f(h) = k$.

- **Vertex Formula**: Given the quadratic $f(x) = ax^2 + bx + c$, the vertex is found using

$$\boxed{\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).}$$

Common Mistakes to Avoid:

- Notice that the maximum or minimum value is the $y-$coordinate of the parabola's vertex. Do not record the extreme value as the $x-$coordinate.

- Determining whether the $y-$coordinate of the vertex is a maximum or minimum depends on whether a is positive or negative. It does NOT depend on whether the $y-$coordinate is positive or negative.

PROBLEMS

1. **Find the maximum or minimum value of each quadratic function. State whether it is a maximum or minimum.**

(a) $f(x) = 3x^2 - 6x + 2$

Here, we know that $a = 3$, $b = -6$ and $c = 2$. Since $a > 0$, we know that the $y-$coordinate of the vertex is a minimum. However, to find the $y-$coordinate of our vertex we first need to find the $x-$coordinate of the vertex by using $x = -\frac{b}{2a}$.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-6}{2(3)} \\ &= -\frac{-6}{6} \\ &= 1. \end{aligned}$$

Now that we have the $x-$coordinate, we can find the $y-$coordinate of the vertex by finding $f(1)$.

$$\begin{aligned} f(1) &= 3(1)^2 - 6(1) + 2 \\ &= 3 - 6 + 2 \\ &= -1. \end{aligned}$$

$$\boxed{\text{Minimum} = -1}$$

(b) $f(x) = -2x^2 + 8x + 9$

Here, we know that $a = -2$, $b = 8$ and $c = 9$. Since $a < 0$, we know that the $y-$coordinate of the vertex is a maximum. However, to find the $y-$coordinate of our vertex we first need to find the $x-$coordinate of the vertex by using $x = -\frac{b}{2a}$.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{8}{2(-2)} \\ &= -\frac{8}{-4} \\ &= 2 \end{aligned}$$

Now that we have the $x-$coordinate, we can find the $y-$coordinate of the vertex by finding $f(2)$.

$$\begin{aligned} f(2) &= -2(2)^2 + 8(2) + 9 \\ &= -2(4) + 16 + 9 \\ &= -8 + 16 + 9 \\ &= 17. \end{aligned}$$

$$\boxed{\text{Maximum} = 17}$$

(c) $f(x) = 2x^2 - 3x + 1$

Here, we know that $a = 2$, $b = -3$ and $c = 1$. Since $a > 0$, we know that the $y-$coordinate of the vertex is a minimum. However, to find the $y-$coordinate of our vertex we first need to find the $x-$coordinate of the vertex by using $x = -\frac{b}{2a}$.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-3}{2(2)} \\ &= -\frac{-3}{4} \\ &= \frac{3}{4} \end{aligned}$$

Now that we have the $x-$coordinate, we can find the $y-$coordinate of the vertex by finding $f\left(\frac{3}{4}\right)$.

$$\begin{aligned} f\left(\frac{3}{4}\right) &= 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 1 \\ &= 2\left(\frac{9}{16}\right) - \frac{9}{4} + 1 \\ &= \frac{9}{8} - \frac{9}{4} + 1 \\ &= \frac{9}{8} - \frac{18}{8} + \frac{8}{8} \\ &= -\frac{1}{8} \end{aligned}$$

$$\boxed{\text{Minimum} = -\frac{1}{8}}$$

(d) $f(x) = -x^2 - 8x + 7$

Here, we have $a = -1$, $b = -8$ and $c = 7$. Since $a < 0$, we know that the $y-$coordinate of the vertex is a maximum. However, to find the $y-$coordinate of our vertex we first need to find the $x-$coordinate of the vertex by using $x = -\frac{b}{2a}$.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-8}{2(-1)} \\ &= -\frac{-8}{-2} \\ &= -4 \end{aligned}$$

Now that we have the $x-$coordinate, we can find the $y-$coordinate of the vertex by finding $f(-4)$.

$$\begin{aligned} f(-4) &= -(-4)^2 - 8(-4) + 7 \\ &= -16 + 32 + 7 \\ &= 23 \end{aligned}$$

$$\boxed{\text{Maximum} = 23}$$

NOTE: you can also solve the above problems by placing the quadratic in standard form and identifying the $y-$coordinate that way.

2. **During the annual frog jumping contest at the county fair, the height of the frog's jump, in feet, is given by**

$$f(x) = -\frac{1}{3}x^2 + \frac{4}{3}x.$$

What was the maximum height reached by the frog?

Since f is a quadratic with $a = -\frac{1}{3} < 0$, the maximum height of the frog's jump is found by identifying the $y-$coordinate of the parabola's vertex. To do this note that $a = -\frac{1}{3}$, $b = \frac{4}{3}$ and $c = 0$. First, we find the $x-$coordinate by using $-\frac{b}{2a}$.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{\frac{4}{3}}{2\left(-\frac{1}{3}\right)} \\ &= -\frac{\frac{4}{3}}{-\frac{2}{3}} \\ &= 2. \end{aligned}$$

Next, we will have the maximum height once we find $f(2)$.

$$\begin{aligned} f(2) &= -\frac{1}{3}(2)^2 + \frac{4}{3}(2) \\ &= -\frac{1}{3}(4) + \frac{8}{3} \\ &= -\frac{4}{3} + \frac{8}{3} \\ &= \frac{4}{3}. \end{aligned}$$

$$\boxed{\text{Maximum height of frog's jump} = \frac{4}{3} \text{ ft}}$$

3. **A ball is thrown directly upward from an initial height of 50 feet. If the height, in feet, of the ball after t seconds is given by**

$$f(t) = -16t^2 + 40t + 50,$$

find the maximum height reached by the ball.

Since f is a quadratic with $a = -16 < 0$, the $f(t)$ coordinate of the parabola's vertex will identify the maximum height. To find the t-coordinate of the vertex, we use $t = -\frac{b}{2a}$ where $a = -16$ and $b = 40$.

$$\begin{aligned} t &= -\frac{b}{2a} \\ &= -\frac{40}{2(-16)} \\ &= -\frac{40}{-32} \\ &= \frac{5}{4}. \end{aligned}$$

To find the maximum, we need to evaluate $f\left(\frac{5}{4}\right)$.

$$\begin{aligned} f\left(\frac{5}{4}\right) &= -16\left(\frac{5}{4}\right)^2 + 40\left(\frac{5}{4}\right) + 50 \\ &= -16\left(\frac{25}{16}\right) + 50 + 50 \\ &= -25 + 100 \\ &= 75. \end{aligned}$$

$$\boxed{\text{Maximum height of ball} = 75 \text{ feet}}$$

4. **A toy rocket is launched of the top of a 150 foot cliff. If the height, in feet, of the rocket t seconds after liftoff is given by**

$$f(t) = -16t^2 + 288t + 150,$$

find the maximum height of the rocket and the time it reaches its maximum height.

Since f is a quadratic with $a = -16 < 0$, the $t-$coordinate of the parabola's vertex will identify the time the rocket reaches its maximum height and the $f(t)$ coordinate of the vertex will give the maximum height. We first find the time t by using $t = -\frac{b}{2a}$ where $a = -16$ and $b = 288$.

$$\begin{aligned} t &= -\frac{b}{2a} \\ &= -\frac{288}{2(-16)} \\ &= -\frac{288}{-32} \\ &= 9. \end{aligned}$$

Next, to find the maximum height, we evaluate $f(9)$.

$$\begin{aligned} f(9) &= -16(9)^2 + 288(9) + 150 \\ &= -16(81) + 2592 + 150 \\ &= -1296 + 2592 + 150 \\ &= 1446. \end{aligned}$$

$$\boxed{\text{Time to reach max height} = 9 \text{ seconds}}$$

$$\boxed{\text{Maximum height of rocket} = 1446 \text{ ft}}$$