MATH 11011 **SOLVING EQUATIONS INVOLVING RATIONAL EXPONENTS** **KSU**

Definition:

- **Rational exponent:** If m and n are positive integers with m/n in lowest terms, then

$$\boxed{a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.}$$

(If n is even then we require $a \geq 0$.) In other words, in a rational exponent, the numerator indicates the power and the denominator indicates the root. For example,

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4.$$

Important Properties:

- To solve $ax^{m/n} + b = c$: First, isolate the variable. Then, raise both sides of the expression to the reciprocal of the exponent since $\left(x^{m/n}\right)^{n/m} = x$. Finally, solve for the variable.
- To solve $ax^{m/n} + bx^{m/2n} + c = 0$: Try to factor and use the zero product property.
- **Zero Product Property**: If a and b are real numbers and $a \cdot b = 0$, then $a = 0$ or $b = 0$.
- Whenever you raise both sides to an even power you must check your "solution" in the original equation. Sometimes extraneous solutions occur.

Common Mistakes to Avoid:

- Although you can raise both **sides** of an equation to the same power without changing the solutions, you can NOT raise each **term** to the same power.
- Remember that whenever you have the even root of a positive number, we get two answers: one positive and one negative. For example, if $x^4 = 16$ then by taking the 4th root of both sides we get $x = 2$ AND $x = -2$. Do NOT forget the negative answer when working with even roots.
- Do NOT attach a $\pm$ when working with odd roots. When you take the odd root of a number, you get only one solution.
- Make sure that the variable is isolated before raising both sides to the same power. For example,

$$\left(4x^{2/3}\right)^{3/2} \neq 4x.$$

PROBLEMS

Solve for x in each of the following equations.

1. $4x^{2/3} - 5 = 11$

First, we will isolate the variable.

$$4x^{2/3} - 5 = 11$$
$$4x^{2/3} = 16$$
$$x^{2/3} = 4$$

Next, we will raise both sides to the 3/2 power.

$$\left(x^{2/3}\right)^{3/2} = (4)^{3/2}$$
$$x = 4^{3/2}$$
$$x = \left(\sqrt{4}\right)^3$$
$$x = (\pm 2)^3$$
$$x = \pm 8$$

$$\boxed{x = 8, \qquad x = -8}$$

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2. $x^{2/3} - 2x^{1/3} - 8 = 0$

Notice that we are unable to isolate the variable. However, we do notice that this is a quadratic-type equation. Therefore,

$$x^{2/3} - 2x^{1/3} - 8 = 0$$
$$\left(x^{1/3} - 4\right)\left(x^{1/3} + 2\right) = 0$$

Setting each factor equal to zero, we obtain

$$x^{1/3} - 4 = 0$$
$$x^{1/3} = 4$$
$$\left(x^{1/3}\right)^3 = 4^3$$
$$x = 64$$

$$x^{1/3} + 2 = 0$$
$$x^{1/3} = -2$$
$$\left(x^{1/3}\right)^3 = (-2)^3$$
$$x = -8$$

$$\boxed{x = 64, \qquad x = -8}$$

OR (for an alternative way)
Letting $u = x^{1/3}$, we get

$$x^{2/3} - 2x^{1/3} - 8 = 0$$
$$u^2 - 2u - 8 = 0$$
$$(u - 4)(u + 2) = 0$$

Setting each factor equal to zero, we obtain

$$u - 4 = 0$$
$$u = 4$$
$$x^{1/3} = 4$$
$$\left(x^{1/3}\right)^3 = 4^3$$
$$x = 64$$

$$u + 2 = 0$$
$$u = -2$$
$$x^{1/3} = -2$$
$$\left(x^{1/3}\right)^3 = (-2)^3$$
$$x = -8$$

$$\boxed{x = 64, \qquad x = -8}$$

3. $x^{4/3} - 16 = 0$

First, we will isolate the variable. Then we will raise both sides to the 3/4 power.

$$x^{4/3} - 16 = 0$$
$$x^{4/3} = 16$$
$$\left(x^{4/3}\right)^{3/4} = 16^{3/4}$$
$$x = \left(\sqrt[4]{16}\right)^3$$
$$x = (\pm 2)^3$$
$$x = \pm 8$$

$$\boxed{x = 8, \qquad x = -8}$$

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4. $3x^{5/3} + 96 = 0$

First, we will isolate the variable.

$$3x^{5/3} + 96 = 0$$
$$3x^{5/3} = -96$$
$$x^{5/3} = -32$$

Next, we will raise both sides to the 3/5 power.

$$\left(x^{5/3}\right)^{3/5} = (-32)^{3/5}$$
$$x = \left(\sqrt[5]{-32}\right)^3$$
$$x = (-2)^3$$
$$x = -8$$

$$\boxed{x = -8}$$

5. $4(x+1)^{1/2} - 5(x+1)^{3/2} + (x+1)^{5/2} = 0$

First, we will factor this expression completely.

$$4(x+1)^{1/2} - 5(x+1)^{3/2} + (x+1)^{5/2} = 0$$
$$(x+1)^{1/2}\left[4 - 5(x+1) + (x+1)^2\right] = 0$$
$$(x+1)^{1/2}\left[4 - 5x - 5 + x^2 + 2x + 1\right] = 0$$
$$(x+1)^{1/2}(x^2 - 3x) = 0$$
$$x(x+1)^{1/2}(x-3) = 0$$

Setting each factor equal to zero, we obtain

$x = 0$	$x - 3 = 0$ $x = 3$	$(x+1)^{1/2} = 0$ $\left[(x+1)^{1/2}\right]^2 = 0^2$ $x + 1 = 0$ $x = -1$

If we check $x = -1$ by substituting back into our original equation, we find that $x = -1$ is a solution.

$$\boxed{x = 0, \qquad x = 3, \qquad x = -1}$$

6. $x^{1/2} + 6 = 5x^{1/4}$

Since we cannot isolate the variable, we will move everything to one side and factor completely.

$$x^{1/2} + 6 = 5x^{1/4}$$
$$x^{1/2} - 5x^{1/4} + 6 = 0$$
$$\left(x^{1/4} - 3\right)\left(x^{1/4} - 2\right) = 0$$

Setting each factor equal to zero, we get

$$x^{1/4} - 3 = 0$$
$$x^{1/4} = 3$$
$$\left(x^{1/4}\right)^4 = 3^4$$
$$x = 81$$

$$x^{1/4} - 2 = 0$$
$$x^{1/4} = 2$$
$$\left(x^{1/4}\right)^4 = 2^4$$
$$x = 16$$

Because we raised both sides to an even power, we must check our answers in the original equation.

Checking: $x = 81$

$$(81)^{1/2} + 6 = 5(81)^{1/4}$$
$$\sqrt{81} + 6 = 5\sqrt[4]{81}$$
$$9 + 6 = 5(3)$$
$$15 = 15\star$$

Checking: $x = 16$

$$(16)^{1/2} + 6 = 5(16)^{1/4}$$
$$\sqrt{16} + 6 = 5\sqrt[4]{16}$$
$$4 + 6 = 5(2)$$
$$10 = 10\star$$

$$\boxed{x = 81, \qquad x = 16}$$

7. $(x-3)^{3/2} = 27$

Notice that the quantity containing the rational exponent is already isolated. Therefore, raising both sides to the 2/3 power, we get

$$(x-3)^{3/2} = 27$$
$$\left[(x-3)^{3/2}\right]^{2/3} = 27^{2/3}$$
$$x - 3 = 27^{2/3}$$
$$x - 3 = \left(\sqrt[3]{27}\right)^2$$
$$x - 3 = (3)^2$$
$$x - 3 = 9$$
$$x = 12$$

$$\boxed{x = 12}$$

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8. $(x-7)^4 = 16$

Notice that although this equation does not contain a rational exponent, to solve it we will raise both sides to the 1/4 power.

$$(x-7)^4 = 16$$
$$\left[(x-7)^4\right]^{1/4} = 16^{1/4}$$
$$x - 7 = 16^{1/4}$$
$$x - 7 = \sqrt[4]{16}$$
$$x - 7 = \pm 2$$
$$x = 7 \pm 2$$

Simplifying this last equation we get $x = 7 + 2 = 9$ and $x = 7 - 2 = 5$.

$$\boxed{x = 9, \qquad x = 5}$$

9. $(x-3)^{2/5} = (4x)^{1/5}$

Notice that we cannot solve this one by factoring. Therefore, we will first eliminate the denominator of the rational exponent by raising both sides to the 5th power.

$$(x-3)^{2/5} = (4x)^{1/5}$$

$$\left[(x-3)^{2/5}\right]^5 = \left[(4x^{1/5}\right]^5$$

$$(x-3)^2 = 4x$$

$$x^2 - 6x + 9 = 4x$$

$$x^2 - 10x + 9 = 0$$

$$(x-9)(x-1) = 0$$

Setting each factor equal to zero, we obtain

$x - 9 = 0$	$x - 1 = 0$
$x = 9$	$x = 1$

$$\boxed{x = 9, \qquad x = 1}$$