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Ablowitz-Ladik hierarchy of integrable equations on a time-space scale

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We derive the Toda's lattice, the Hirota's network, and the nonlinear Schrodinger dynamic equations on a time-space scale by extension on a time-space scale the Ablowitz-Ladik hierarchy of integrable dynamic systems. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4896564]

I. INTRODUCTION

There are several well-known methods to generate integrable physically important nonlinear dynamic systems. The most efficient ones are the Ablowitz-Kaup-Newel-Segur (AKNS) method[1] and Ablowitz-Ladik (AL) method,[2] that are based on the Zakharov and Shabat spectral problem.[11] Ablowitz-Ladik hierarchy is a set of integrable discrete nonlinear dynamic systems. It is known that Ablowitz-Ladik hierarchy has a variety of modeling applications including optics,[3,4] chaos in dispersive numerical schemes,[7] and so on.

In Ref. 8 Hilger introduced the time scale calculus that unifies continuous and discrete analysis. The time scales calculus was further developed in the books.[5,6]

In this paper we extend the Ablowitz-Ladik hierarchy of nonlinear dynamic systems on a time-space scale. Note that a time-space scale introduced here contains partial difference-differential equations with variable graininess. We hope that the time-space scale extension of Ablowitz-Ladik hierarchy will give a wider range of integrable dynamic systems that could be used in modeling.

II. ABLOWITZ-LADIK NABLA DYNAMIC SYSTEMS

Let $\mathbb{T}$ and $\mathbb{X}$ be an arbitrary nonempty closed subsets of real numbers (time and space scales). For $t \in \mathbb{T}$ and $x \in \mathbb{X}$ we define backward jump operators $\sigma : \mathbb{T} \to \mathbb{T}$, $\rho : \mathbb{X} \to \mathbb{X}$:

$$\sigma(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \rho(x) := \sup\{y \in \mathbb{X} : y < x\}. \tag{2.1}$$

For $x \in \mathbb{X}$ we also define the forward jump operator $\beta(x) : \mathbb{X} \to \mathbb{X}$ by

$$\beta(x) = \rho^{-1}(x) := inf\{y \in \mathbb{T} : y > x\}. \tag{2.2}$$

We are considering the nabla derivatives (see Refs. 5 and 6) (instead of delta derivatives (see Ref. 8), since in physics applications nabla derivatives with respect to the time variable are casual.

By definition of nabla derivatives with respect to time (t) and space (x) variables,[6]

$$v^{(x)}(t,x) := v^{\nabla_x}(t,x) = \lim_{q \searrow \nu(x)} \frac{v(t,x) - v(t,\rho(x))}{q}, \tag{2.3}$$

$$v^{(t)}(t,x) := v^{\nabla_t}(t,x) = \lim_{p \searrow \mu(t)} \frac{v(t,x) - v(\sigma(t),x)}{p}, \tag{2.4}$$

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where the graininess functions $\mu : \mathbb{T} \to [0, \infty)$, $\nu : \mathbb{X} \to [0, \infty)$ are defined as

$$\mu(t) = t - \sigma(t), \quad \nu(x) = x - \rho(x). \tag{2.5}$$

We are going to use the following notation:

$$f^\rho = f^\rho(t, x) := f(t, \rho(x)), \quad f^\sigma = f^\sigma(t, x) := f(\sigma(t), x),$$

$$f^{\rho^{-1}} = f^{\rho^{-1}}(t, x) = f^\beta(t, x) := f(t, \beta(x)) = f(t, \rho^{-1}(x)). \tag{2.6}$$

Note that

$$f^{\sigma\rho}(t, x) = f^{\rho\sigma}(t, x) = f(\sigma(t), \rho(x)), \tag{2.7}$$

$$f^\rho = f(t, x) - \nu(x) f^{(x)}(t, x), \quad f^\sigma = f(t, x) - \mu(t) v^{(t)}(t, x). \tag{2.8}$$

Lemma 2.1. If the functions f(t, x), $\mu(t)$, $\nu(x)$ are nabla differentiable (see Ref. 6) and

$$\mu^{(x)}(t) = \nu^{(t)}(x) = 0, \tag{2.9}$$

then

$$f^{(tx)}(t, x) = f^{(xt)}(t, x), \quad f^{\rho(t)}(t, x) = f^{(t)\rho}(t, x). \tag{2.10}$$

Consider nabla dynamic systems of the form

$$v^\rho(t, x) = M(t, x) v(t, x), \tag{2.11}$$

$$v^{(t)}(t, x) = N(t, x) v(t, x), \tag{2.12}$$

where

$$v(t, x) = \begin{pmatrix} v_1(t, x) \\ v_2(t, x) \end{pmatrix}, \quad N(t, x) = \begin{pmatrix} A(t, x) & B(t, x) \\ C(t, x) & D(t, x) \end{pmatrix}, \tag{2.13}$$

$$M(t, x) = \frac{1}{1 - S(t, x)T(t, x)} \begin{pmatrix} R(t, x)S(t, x) + \Lambda_1(\zeta) & Q(t, x) + \Lambda_2(\zeta)S(t, x) \\ R(t, x) + T(t, x)\Lambda_1(\zeta) & Q(t, x)T(t, x) + \Lambda_2(\zeta) \end{pmatrix}, \tag{2.14}$$

$A(t, x)$, $B(t, x)$, $C(t, x)D(t, x)$, $R(t, x)$, $Q(t, x)$, $S(t, x)$, $T(t, x)$, $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$ are given functions, and ζ is a spectral parameter.

To derive the integrable nonlinear equations with respect to the potentials $R(t, x)$, $Q(t, x)$, $S(t, x)$, $T(t, x)$ from the linear systems (2.11), (2.12) one can use (see Refs. 1 and 2) condition (2.10), and the spectral expansion:

$$A(t, x) = A_1(t, x)\zeta + A_0(t, x), \quad B(t, x) = B_{-1}(t, x)\zeta^{-1} + B_0(t, x),$$

$$C(t, x) = C_1(t, x)\zeta + C_0(t, x), \quad D(t, x) = D_{-1}(t, x)\zeta^{-1} + D_0(t, x), \tag{2.15}$$

$$\Lambda_1 = \zeta, \quad \Lambda_2 = \frac{1}{\zeta}, \tag{2.16}$$

where ζ are invariant eigenvalues of the spectral problem (2.11). Note that it is possible to get more general nonlinear dynamic systems by considering polynomial expansion rather than (2.16) and the time dependent spectral parameter ζ.

By using (2.10), (2.15)–(2.16) one can derive from (2.11)–(2.14) the following time evolution equations (see Sec. III) for the four functions $S(t, x)$, $T(t, x)$, $R(t, x)$, $Q(t, x)$ (here and further we often suppress the (t, x) to shorten the formulas):

$$S^{(t)}(t, x) = A_0 S - D_0 S^\sigma + A_1 Q^\rho (1 - ST) + D_{-1} S^\sigma QT - A_1 RS^2 +$$

$$A_1 T^{\sigma\beta} Q^\sigma S - D_{-1} S^{\sigma\beta} (RS)^\sigma + A_1 R^\sigma S^\sigma S - D_{-1} Q^\sigma, \tag{2.17}$$

$$T^{(t)}(t,x) = D_0 T - A_0 T^\sigma + D_{-1} R^\rho (1 - ST) + A_1 RST^\sigma - D_{-1} QT^2 +$$
$$D_{-1} TS^{\sigma\beta} R^\sigma - A_1 T^{\sigma\beta} Q^\sigma T^\sigma + D_{-1} Q^\sigma T^\sigma T - A_1 R^\sigma, \tag{2.18}$$

$$R^{(t)}(t,x) = RD_0 - R^\sigma A_0 + D_{-1} T(1 - RQ) + D_{-1} R^\sigma RS^{\sigma\beta} - A_1 T^{\sigma\beta}, \tag{2.19}$$

$$Q^{(t)}(t,x) = QA_0 - Q^\sigma D_0 + A_1 S(1 - RQ) + A_1 Q^\sigma QT^{\sigma\beta} - D_{-1} S^{\sigma\beta}, \tag{2.20}$$

where $A_0(t, x)$, $D_0(t, x)$ are arbitrary functions if the time and the space variables are continuous ($\nu(x) \equiv \mu(t) \equiv 0$), and additionally $(ST)^{(t)} \equiv 0$. Otherwise the functions $A_0(t, x)$, $D_0(t, x)$ are defined as follows:

$$A_0(t,x) = \begin{cases} \frac{1-ST}{(ST)^\sigma - ST}\left[A_1\left(QT - (QT)^\sigma + RS - (RS)^\sigma\right) + \frac{(1-ST)^{(t)}}{1-ST}\right], & \nu(x) \equiv 0, \\ c_1 e_K(t,x,x_0) + \int_{x_0}^x e_K(t,x,\rho(y)) f(t,y) \nabla y, & \nu(x) \neq 0, \end{cases} \tag{2.21}$$

$$D_0(t,x) = \begin{cases} \frac{1-ST}{(ST)^\sigma - ST}\left[D_{-1}\left(RS - (SR)^\sigma + QT - (QT)^\sigma\right) + \frac{(1-ST)^{(t)}}{1-ST}\right], & \nu(x) \equiv 0, \\ c_2 e_K(t,x,x_0) + \int_{x_0}^x e_K(t,x,\rho(y)) g(t,y) \nabla y, & \nu(x) \neq 0, \end{cases} \tag{2.22}$$

where c_1, c_2 are the constants, $e_K(t, x, y)$ is the nabla exponential function on a space scale (see Refs. 8 and 6):

$$e_K(t,x,y) = \exp\left(\int_{x_0}^x \lim_{q \searrow \nu(y)} \frac{Log(1 - qK(t,y))}{-q} \nabla y\right), \tag{2.23}$$

$$K(t,x) = \frac{S(t,x)T(t,x) - (ST)^\sigma(t,x)}{\nu(x)(1 - (ST)^\sigma(t,x))}, \tag{2.24}$$

$$f(t,x) = \frac{1 - ST}{\nu(x)(1 - (ST)^\sigma)}\left(A_1[Q^\rho T - T^{\sigma\beta} Q^\sigma + RS - (RS)^\sigma] + \frac{(1 - ST)^{(t)}}{1 - ST}\right), \tag{2.25}$$

$$g(t,x) = \frac{1 - ST}{\nu(x)(1 - (ST)^\sigma)}\left(D_{-1}[R^\rho S - S^{\sigma\beta} R^\sigma + QT - (QT)^\sigma] + \frac{(1 - ST)^{(t)}}{1 - ST}\right). \tag{2.26}$$

To simplify the time evolution equations (2.17)–(2.20) for the four unknown functions $S(t, x)$, $T(t, x)$, $R(t, x)$, $Q(t, x)$ consider the following particular cases.

The first case. Choosing

$$R(t,x) \equiv 0, \quad T(t,x) \equiv 1, \tag{2.27}$$

from (2.19) and (2.18) we get

$$A_1(t,x) = D_{-1}(t,x), \tag{2.28}$$

$$D_0(t,x) = A_0(t,x) + A_1(t,x)Q(t,x). \tag{2.29}$$

Further from (2.21) where $\nu(x) \neq 0$ we get

$$A_0^{(x)} - KA_0 = f = \frac{1 - S}{\nu(1 - S^\sigma)} A_1 \left(Q^\rho - Q^\sigma + \frac{(1 - S)^{(t)}}{1 - S}\right),$$
$$\frac{A_0 - A_0^\rho}{\nu} - \frac{S - S^\sigma}{\nu(1 - S^\sigma)} A_0 = \frac{1 - S}{\nu(1 - S^\sigma)} A_1 \left(Q^\rho - Q^\sigma + \frac{(1 - S)^{(t)}}{1 - S}\right),$$

or

$$A_0 - A_0^\rho \frac{1 - S^\sigma}{1 - S} = A_1(Q^\rho - Q^\sigma) + \frac{(1 - S)^{(t)}}{1 - S}.$$

In the same way from (2.22) where $\nu(x) \neq 0$ we get

$$D_0 - D_0^\rho \frac{1 - S^\sigma}{1 - S} = D_{-1}(Q - Q^\sigma) + \frac{(1 - S)^{(t)}}{1 - S}.$$

By subtracting the above two formulas we get

$$A_0 - D_0 + (D_0 - A_0)^\rho \frac{1 - S^\sigma}{1 - S} = A_1(Q^\rho - Q),$$

hence using (2.29) we have

$$A_1(t,x) = \frac{(1 - S^\sigma(t,x))A_1^\rho(t,x)}{1 - S(t,x)}. \tag{2.30}$$

Furthermore the evolution equations (2.17), (2.20):

$$S^{(t)}(t,x) = A_0 S - D_0 S^\sigma + A_1 Q^\rho(1 - S) + A_1(QS^\sigma + Q^\sigma S - Q^\sigma) =$$

$$A_0 S - S^\sigma(A_0 + A_1 Q) + A_1 S(Q^\sigma - Q^\rho) + A_1(Q^\rho + S^\sigma Q - Q^\sigma),$$

$$Q^{(t)}(t,x) = A_0 Q - D_0 Q^\sigma + A_1(S - S^{\sigma\beta} + Q^\sigma Q)$$

are simplified to the nonlinear dynamic system for two functions $S(t, x)$, $Q(t, x)$:

$$S^{(t)}(t,x) = A_0(S - S^\sigma) + A_1(Q^\rho - Q^\sigma)(1 - S), \tag{2.31}$$

$$Q^{(t)}(t,x) = A_0(Q - Q^\sigma) + A_1(S - S^{\sigma\beta}). \tag{2.32}$$

In the case of the continuous time scale

$$\mu(t) = 0, \quad \nu(x) \neq 0, \tag{2.33}$$

(2.31) becomes

$$\partial_t S(t,x) = A_1(t,x)(1 - S(t,x))(Q^\rho(t,x) - Q(t,x)).$$

Solving for $S(t, x)$ we get the Toda's lattice on a space scale (Ref. 10):

$$\partial_t Q(t,x) = A_1(t,x)(S(t,x) - S^\beta(t,x)), \quad S(t,x) = 1 + Ce^{\int_{t_0}^t A_1(s,x)(Q(s,x) - Q^\rho(s,x))ds}. \tag{2.34}$$

The second case. Choosing

$$R(t,x) = Q(t,x), \quad S(t,x) = T(t,x), \quad A_1(t,x) = D_{-1}(t,x), \quad A_0(t,x) = D_0(t,x), \tag{2.35}$$

we get from (2.17), (2.19) the nonlinear dynamic system

$$R^{(t)}(t,x) = A_0(R - R^\sigma) + A_1 S(1 - R^2) - A_1 S^{\sigma\beta}(1 - R^\sigma R), \tag{2.36}$$

$$S^{(t)}(t,x) = (A_0 - A_1 RS + A_1 S^{\sigma\beta} R^\sigma)(S - S^\sigma) + A_1 R^\rho(1 - S^2) - A_1 R^\sigma(1 - S^\sigma S). \tag{2.37}$$

In the case of the continuous time scale we get the nonlinear self-dual network on a space scale (proposed by Hirota[9]):

$$\partial_t R(t,x) = A_1(t,x)(1 - R^2(t,x))(S(t,x) - S^\beta(t,x)),$$

$$\partial_t S(t,x) = A_1(t,x)(1 - S^2(t,x))(R^\rho(t,x) - R(t,x)). \tag{2.38}$$

The third case. In the case of the continuous time scale from (2.21), (2.22) we get

$$A_0(t,x) = c_1 + \int_{x_0}^x \frac{(1 - ST)^{(t)}}{1 - ST} \frac{\nabla x}{\nu(x)} - A_1 Q T^\beta,$$

$$D_0(t,x) = c_2 + \int_{x_0}^x \frac{(1 - ST)^{(t)}}{1 - ST} \frac{\nabla x}{\nu(x)} - D_{-1} R S^\beta, \tag{2.39}$$

$$A_0(t,x) - D_0(t,x) = D_{-1} R S^\beta - A_1 Q T^\beta. \tag{2.40}$$

Further from (2.17)–(2.20) we have

$$\partial_t S(t,x) = (A_0 - D_0)S + (A_1 Q^\rho - D_{-1}Q)(1-ST) + A_1 T^\beta QS - D_{-1}S^\beta RS,$$

$$\partial_t T(t,x) = (D_0 - A_0)T + (D_{-1}R^\rho - A_1 R)(1-ST) + D_{-1}TS^\beta R - A_1 T^\beta QT,$$

$$\partial_t R(t,x) = R(D_0 - A_0) + D_{-1}T(1-RQ) + D_{-1}R^2 S^\beta - A_1 T^\beta,$$

$$\partial_t Q(t,x) = Q(A_0 - D_0) + A_1 S(1-RQ) + A_1 Q^2 T^\beta - D_{-1}S^\beta,$$

and using (2.40) we get

$$\partial_t S(t,x) = (A_1 Q^\rho(t,x) - D_{-1}Q(t,x))(1 - S(t,x)T(t,x)),$$

$$\partial_t R(t,x) = (D_{-1}T(t,x) - A_1 T^\beta(t,x))(1 - R(t,x)Q(t,x)),$$

$$\partial_t T(t,x) = (D_{-1}R^\rho(t,x) - A_1 R(t,x))(1 - S(t,x)T(t,x)),$$

$$\partial_t Q(t,x) = (A_1 S(t,x) - D_{-1}S^\beta(t,x))(1 - R(t,x)Q(t,x)). \tag{2.41}$$

Further choosing

$$R(t,x) = \overline{Q}(t,x), \quad S(t,x) = \overline{T}(t,x), \quad A_1 = \overline{D_{-1}}, \tag{2.42}$$

we get the nonlinear Schrodinger dynamic system on a space scale:

$$\partial_t Q(t,x) = (A_1 S(t,x) - \overline{A_1} S^\beta(t,x))(1 - Q(t,x)\overline{Q}(t,x)),$$

$$\partial_t S(t,x) = (A_1 Q^\rho(t,x) - \overline{A_1} Q(t,x))(1 - S(t,x)\overline{S}(t,x)). \tag{2.43}$$

Furthermore in the case $S(t, x) \equiv 1$ we get the nonlinear Schrodinger equation on the continuous time-space scale

$$\partial_t Q(t,x) = (A_1 - \overline{A_1})(1 - Q\overline{Q}). \tag{2.44}$$

III. PROOFS

Proof of Lemma 2.1. Note that from the conditions of Lemma 2.1 we get

$$\mu^\rho(t) = \mu(t) - \nu(x)\mu^{(x)}(t) = \mu(t), \quad \nu^\sigma(x) = \nu(x) - \mu(t)\nu^{(t)}(x) = \nu(x).$$

By the definition of the nabla partial derivatives in the case $\mu(t) \neq 0$, $\nu(x) \neq 0$ we get

$$f^{(tx)} - f^{(xt)} = \frac{\frac{f-f^\sigma}{\mu(t)} - \left(\frac{f-f^\sigma}{\mu(t)}\right)^\rho}{\nu(x)} - \frac{\frac{f-f^\rho}{\nu(x)} - \left(\frac{f-f^\rho}{\nu(x)}\right)^\sigma}{\mu(t)} =$$
$$\frac{f - f^\sigma - f^\rho + f^{\sigma\rho}}{\mu(t)\nu(x)} - \frac{f - f^\rho - f^\sigma + f^{\rho\sigma}}{\mu(t)\nu(x)} = 0.$$

Thus we have

$$f^{\rho t}(t,x) = \left(f(t,x) - \nu(x)f^{(x)}\right)^{(t)} = f^{(t)}(t,x) - \nu(x)f^{(xt)} = f^{t\rho}(t,x).$$

In other cases (2.10) is proved similarly. □

Rewrite the systems (2.11)–(2.12) in the form

$$v_1^\rho(t,x) = \frac{R(t,x)S(t,x) + \Lambda_1(\zeta)}{1 - S(t,x)T(t,x)} v_1(t,x) + \frac{Q(t,x) + \Lambda_2(\zeta)S(t,x)}{1 - S(t,x)T(t,x)} v_2(t,x), \tag{3.1}$$

$$v_2^\rho(t,x) = \frac{R(t,x)+T(t,x)\Lambda_1(\zeta)}{1-S(t,x)T(t,x)}v_1(t,x) + \frac{Q(t,x)T(t,x)+\Lambda_2(\zeta)}{1-S(t,x)T(t,x)}v_2(t,x), \tag{3.2}$$

$$v_1^{(t)}(t,x) = A(t,x)v_1(t,x) + B(t,x)v_2(t,x), \tag{3.3}$$

$$v_2^{(t)}(t,x) = C(t,x)v_1(t,x) + D(t,x)v_2(t,x). \tag{3.4}$$

Multiplying Eqs. (3.1), (3.2) by $1 - S(t,x)T(t,x)$ and (nabla) differentiating with respect to t we get

$$((1-ST)v_1^\rho)^{(t)} = ((RS+\Lambda_1)v_1)^{(t)}(+(Q+S\Lambda_2)v_2)^{(t)},$$

$$((1-ST)v_2^\rho)^{(t)} = ((R+T\Lambda_1)v_1)^{(t)} + ((QT+\Lambda_2)\,v_2)^{(t)}.$$

By using the time-space scale product rules (see Ref. 6)

$$(f(t,x)g(t,x))^{(t)} = f^{(t)}(t,x)g(t,x) + f^\sigma(t,x)g^{(t)}(t,x), \tag{3.5}$$

$$(f(t,x)g(t,x))^{(x)} = f^{(x)}(t,x)g(t,x) + f^\rho(t,x)g^{(x)}(t,x), \tag{3.6}$$

we have

$$(1-ST)^\sigma v_1^{\rho t} + (1-ST)^{(t)}v_1^\rho =$$

$$(RS+\Lambda_1)^\sigma v_1^{(t)} + (RS+\Lambda_1)^{(t)}v_1 + (Q+S\Lambda_2)^\sigma v_2^{(t)} + (Q+S\Lambda_2)^{(t)}v_2, \tag{3.7}$$

$$(1-ST)^\sigma v_2^{\rho t} + (1-ST)^{(t)}v_2^\rho =$$

$$(R+T\Lambda_1)^\sigma v_1^{(t)} + (R+T\Lambda_1)^{(t)}v_1 + (QT+\Lambda_2)^\sigma v_2^{(t)} + (QT+\Lambda_2)^{(t)}v_2. \tag{3.8}$$

Further by using Lemma 2.1 from (3.3) and (3.4) we get compatibility conditions

$$\left(v_1^\rho\right)^{(t)} = v_1^{\rho(t)} = v_1^{(t)\rho} = \left(v_1^{(t)}\right)^\rho = A^\rho v_1^\rho + B^\rho v_2^\rho,$$

$$\left(v_2^\rho\right)^{(t)} = v_2^{\rho(t)} = v_2^{(t)\rho} = \left(v_2^{(t)}\right)^\rho = C^\rho v_1^\rho + D^\rho v_2^\rho, \tag{3.9}$$

and by substitution in (3.7), (3.8) we have

$$(1-ST)^\sigma(A^\rho v_1^\rho + B^\rho v_2^\rho) + (1-ST)^{(t)}v_1^\rho = (RS+\Lambda_1)^\sigma(Av_1+Bv_2)+$$

$$+(RS+\Lambda_1)^{(t)}v_1 + (Q+S\Lambda_2)^\sigma(Cv_1+Dv_2) + (Q+S\Lambda_2)^{(t)}v_2, \tag{3.10}$$

$$(1-ST)^\sigma(C^\rho v_1^\rho + D^\rho v_2^\rho) + (1-ST)^{(t)}v_2^\rho = (R+T\Lambda_1)^\sigma(Av_1+Bv_2)+$$

$$+(R+T\Lambda_1)^{(t)}v_1 + (QT+\Lambda_2)^\sigma(Cv_1+Dv_2) + (QT+\Lambda_2)^{(t)}v_2. \tag{3.11}$$

Further using (3.1) we replace v_j^ρ:

$$[(1-ST)^\sigma A^\rho + (1-ST)^{(t)}]\left(\frac{RS+\Lambda_1}{1-ST}v_1 + \frac{Q+\Lambda_2 S}{1-ST}v_2\right)$$

$$+(1-ST)^\sigma B^\rho\left(\frac{R+T\Lambda_1}{1-ST}v_1 + \frac{QT+\Lambda_2}{1-ST}v_2\right) = (RS+\Lambda_1)^\sigma(Av_1+Bv_2)$$

$$+(RS+\Lambda_1)^{(t)}v_1 + (Q+S\Lambda_2)^\sigma(Cv_1+Dv_2) + (Q+S\Lambda_2)^{(t)}v_2, \tag{3.12}$$

$$(1-ST)^{\sigma}C^{\rho}\left(\frac{RS+\Lambda_1}{1-ST}v_1+\frac{Q+\Lambda_2 S}{1-ST}v_2\right)$$

$$+[(1-ST)^{\sigma}D^{\rho}+(1-ST)^{(t)}]\left(\frac{R+T\Lambda_1}{1-ST}v_1+\frac{QT+\Lambda_2}{1-ST}v_2\right)$$

$$=(R+T\Lambda_1)^{\sigma}(Av_1+Bv_2)$$

$$+(R+T\Lambda_1)^{(t)}v_1+(QT+\Lambda_2)^{\sigma}(Cv_1+Dv_2)+(QT+\Lambda_2)^{(t)}v_2. \tag{3.13}$$

By equating the coefficients of $v_j(t,x)$, $j=1,2$ we get the system

$$[(1-ST)^{\sigma}A^{\rho}+(1-ST)^{(t)}]\left(\frac{RS+\Lambda_1}{1-ST}\right)+(1-ST)^{\sigma}B^{\rho}\left(\frac{R+T\Lambda_1}{1-ST}\right)$$

$$=(RS+\Lambda_1)^{\sigma}A+(RS+\Lambda_1)^{(t)}+(Q+S\Lambda_2)^{\sigma}C, \tag{3.14}$$

$$(1-ST)^{\sigma}C^{\rho}\left(\frac{RS+\Lambda_1}{1-ST}\right)+[(1-ST)^{\sigma}D^{\rho}+(1-ST)^{(t)}]\left(\frac{R+T\Lambda_1}{1-ST}\right)$$

$$=(R+T\Lambda_1)^{\sigma}A+(R+T\Lambda_1)^{(t)}+(QT+\Lambda_2)^{\sigma}C, \tag{3.15}$$

$$[(1-ST)^{\sigma}A^{\rho}+(1-ST)^{(t)}]\left(\frac{Q+\Lambda_2 S}{1-ST}\right)+(1-ST)^{\sigma}B^{\rho}\left(\frac{QT+\Lambda_2}{1-ST}\right)$$

$$=(RS+\Lambda_1)^{\sigma}B+(Q+S\Lambda_2)^{\sigma}D+(Q+S\Lambda_2)^{(t)}, \tag{3.16}$$

$$(1-ST)^{\sigma}C^{\rho}\left(\frac{Q+\Lambda_2 S}{1-ST}\right)+[(1-ST)^{\sigma}D^{\rho}+(1-ST)^{(t)}]\left(\frac{QT+\Lambda_2}{1-ST}\right)$$

$$=(R+T\Lambda_1)^{\sigma}B+(QT+\Lambda_2)^{\sigma}D+(QT+\Lambda_2)^{(t)}. \tag{3.17}$$

Considering the spectral expansion

$$A(t,x)=A_1(t,x)\zeta+A_0(t,x),\quad B(t,x)=B_{-1}(t,x)\zeta^{-1}+B_0(t,x),$$

$$C(t,x)=C_1(t,x)\zeta+C_0(t,x),\quad D(t,x)=D_{-1}(t,x)\zeta^{-1}(t,x)+D_0(t,x), \tag{3.18}$$

$$\Lambda_1=\zeta,\quad \Lambda_2=\frac{1}{\zeta}, \tag{3.19}$$

where ζ are invariant eigenvalues we get 4 equations for four unknown functions $S(t,x)$, $T(t,x)$, $R(t,x)$, $Q(t,x)$,

$$[(1-ST)^{\sigma}(A_1\zeta+A_0)^{\rho}+(1-ST)^{(t)}]\left(\frac{RS+\zeta}{1-ST}\right)$$

$$+(1-ST)^{\sigma}(B_{-1}\zeta^{-1}+B_0)^{\rho}\left(\frac{R+T(\zeta)}{1-ST}\right)=$$

$$=(RS+\zeta)^{\sigma}(A_1\zeta+A_0)+(RS+\zeta)^{(t)}+(Q+S(\frac{1}{\zeta}))^{\sigma}(C_1\zeta+C_0), \tag{3.20}$$

$$(1-ST)^{\sigma}(C_2\zeta^2+C_1\zeta+C_0)^{\rho}\frac{RS+\zeta}{1-ST}$$
$$+[(1-ST)^{\sigma}(D_{-1}\zeta^{-1}+D_0)^{\rho}+(1-ST)^{(t)}]\left(\frac{R+T(\zeta)}{1-ST}\right)$$
$$=(R+T(\zeta))^{\sigma}(A_1\zeta+A_0)+(R+T(\zeta))^{(t)}+(QT+\frac{1}{\zeta})^{\sigma}(C_1\zeta+C_0), \tag{3.21}$$

$$[(1-ST)^{\sigma}(A_1\zeta+A_0)^{\rho}+(1-ST)^{(t)}]\left(\frac{Q+(\frac{1}{\zeta})S}{1-ST}\right)$$
$$+(1-ST)^{\sigma}(B_{-1}\zeta^{-1}+B_0)^{\rho}\left(\frac{QT+\frac{1}{\zeta}}{1-ST}\right)=(RS+\zeta)^{\sigma}(B_{-1}\zeta^{-1}+B_0)$$
$$+(Q+S(\frac{1}{\zeta}))^{\sigma}(D_{-1}\zeta^{-1}+D_0)+\left(Q+S(\frac{1}{\zeta})\right)^{(t)}, \tag{3.22}$$

$$(1-ST)^{\sigma}\,(C_1\zeta+C_0)^{\rho}\,\frac{Q+(\frac{1}{\zeta})S}{1-ST}+[(1-ST)^{\sigma}(D_{-1}\zeta^{-1}+D_0)^{\rho}$$
$$+(1-ST)^{(t)}]\frac{QT+\frac{1}{\zeta}}{1-ST}=(R+T(\zeta))^{\sigma}(B_{-1}\zeta^{-1}+B_0)$$
$$+(QT+\frac{1}{\zeta})^{\sigma}(D_{-1}\zeta^{-1}+D_0)+\left(QT+\frac{1}{\zeta}\right)^{(t)}. \tag{3.23}$$

Equations (3.20)–(3.23) yield a sequence of equations corresponding to the powers of ζ^k, $k=\pm 2$, ± 1, 0 all of which must be independently satisfied. By solving the equations corresponding to the powers of ζ (starting with the highest and the lowest powers) we get

$$D_{-1}^{\rho}=\frac{(1-ST)}{(1-ST)^{\sigma}}D_{-1},\quad B_{-1}^{\rho}=\frac{(1-ST)S^{\sigma}}{(1-ST)^{\sigma}}D_{-1}=S^{\sigma}D_{-1}^{\rho}, \tag{3.24}$$

$$A_1^{\rho}=\frac{1-ST}{(1-ST)^{\sigma}}A_1,\quad C_1^{\rho}=\frac{(1-ST)T^{\sigma}}{(1-ST)^{\sigma}}A_1,\quad C_0=D_{-1}R, \tag{3.25}$$

$$D_0-D_0^{\rho}\frac{(1-ST)^{\sigma}}{1-ST}=D_{-1}[(R)^{\rho}S-S^{\sigma\rho^{-1}}(R)^{\sigma}+(QT)-(QT)^{\sigma}]+\frac{(1-ST)^{(t)}}{1-ST}, \tag{3.26}$$

$$A_0-A_0^{\rho}\frac{(1-ST)^{\sigma}}{1-ST}=A_1[Q^{\rho}T-T^{\sigma\rho^{-1}}Q^{\sigma}+RS-(RS)^{\sigma}]+\frac{(1-ST)^{(t)}}{1-ST}, \tag{3.27}$$

$$B_0=A_1^{\rho}(Q)]\frac{(1-ST)^{\sigma}}{1-ST}=QA_1,\quad C_1=A_1^{\rho}T^{\sigma}, \tag{3.28}$$

and the time evolution equations

$$S^{(t)}=[A_0^{\rho}S+B_{-1}^{\rho}(QT)+B_0^{\rho}]\frac{(1-ST)^{\sigma}}{1-ST},$$
$$+\frac{(1-ST)^{(t)}S}{1-ST}-B_{-1}(RS)^{\sigma}-SD_0^{\sigma}-D_{-1}(Q)^{\sigma}, \tag{3.29}$$

$$T^{(t)}=A_1[T^{\sigma}RS-(QT)^{\sigma}T^{\sigma\rho^{-1}}-R^{\sigma}]+D_{-1}R^{\rho}$$
$$+TD_0-T^{\sigma}A_0-TD_{-1}[R^{\rho}S-S^{\sigma\rho^{-1}}R^{\sigma}+QT-(QT)^{\sigma}], \tag{3.30}$$

$$(RS)^{(t)}=(A_0-A_1RS)(RS-(RS)^{\sigma})+D_{-1}^{\rho}S^{\sigma}T-D_{-1}Q^{\sigma}R$$
$$+A_1[Q^{\rho}R-S^{\sigma}T^{\sigma\rho^{-1}}]+A_1RS(T^{\sigma\rho^{-1}}Q^{\sigma}-Q^{\rho}T), \tag{3.31}$$

$$R^{(t)}=RD_0-R^{\sigma}A_0-A_1T^{\sigma\rho^{-1}}+D_{-1}T+D_{-1}R(R^{\sigma}S^{\sigma\rho^{-1}}-QT), \tag{3.32}$$

$$Q^{(t)} = QA_0 - Q^\sigma D_0 + A_1 S(1 - RQ) + A_1 Q^\sigma Q T^{\sigma\rho^{-1}} - D_{-1} S^{\sigma\rho^{-1}}, \tag{3.33}$$

$$(QT)^{(t)} = (D_0 - D_{-1}QT)(QT - (QT)^\sigma) + A_1^\rho T^\sigma S - A_1 Q R^\sigma$$
$$+D_{-1}[R^\rho Q - T^\sigma S^{\sigma\rho^{-1}}] + D_{-1}QT(S^{\sigma\rho^{-1}} R^\sigma - R^\rho S). \tag{3.34}$$

Eventually we get the time evolution equations for the functions $S(t, x)$, $T(t, x)$, $R(t, x)$, $Q(t, x)$,

$$S^{(t)} = A_0 S - D_0 S^\sigma + A_1 Q^\rho (1 - ST) + D_{-1} S^\sigma QT - A_1 RS^2 +$$
$$A_1 T^{\sigma\rho^{-1}} Q^\sigma S - D_{-1} S^{\sigma\rho^{-1}} (RS)^\sigma + A_1 R^\sigma S^\sigma S - D_{-1} Q^\sigma, \tag{3.35}$$

$$T^{(t)} = D_0 T - A_0 T^\sigma + D_{-1} R^\rho (1 - ST) + A_1 RST^\sigma - D_{-1} QT^2 +$$
$$D_{-1} T S^{\sigma\rho^{-1}} R^\sigma - A_1 T^{\sigma\rho^{-1}} Q^\sigma T^\sigma + D_{-1} Q^\sigma T^\sigma T - A_1 R^\sigma, \tag{3.36}$$

$$R^{(t)} = RD_0 - R^\sigma A_0 + D_{-1} T(1 - RQ) + D_{-1} R^\sigma R S^{\sigma\rho^{-1}} - A_1 T^{\sigma\rho^{-1}}, \tag{3.37}$$

$$Q^{(t)} = QA_0 - Q^\sigma D_0 + A_1 S(1 - RQ) + A_1 Q^\sigma Q T^{\sigma\rho^{-1}} - D_{-1} S^{\sigma\rho^{-1}}, \tag{3.38}$$

where $A_0(t, x)$, $D_0(t, x)$ satisfy the conditions

$$(A_0 - A_0^\rho)\frac{(1 - ST)^\sigma}{1 - ST} + A_0\left(\frac{(ST)^\sigma - ST}{1 - ST}\right)$$
$$= A_1[Q^\rho T - T^{\sigma\rho^{-1}} Q^\sigma + RS - (RS)^\sigma] + \frac{(1 - ST)^{(t)}}{1 - ST}, \tag{3.39}$$

$$(D_0 - D_0^\rho)\frac{(1 - ST)^\sigma}{1 - ST} + D_0\left(\frac{(ST)^\sigma - ST}{1 - ST}\right)$$
$$= D_{-1}[R^\rho S - S^{\sigma\rho^{-1}} R^\sigma + QT - (QT)^\sigma] + \frac{(1 - ST)^{(t)}}{1 - ST}. \tag{3.40}$$

Note that (3.31), (3.34) are followed from (3.35)–(3.38) and the product rule for the time-scale derivative.

Conditions (3.39)–(3.40) are linear dynamic equations on a time scale with respect to $A_0(t, x)$, $D_0(t, x)$, and they may be solved explicitly. Indeed if $\nu(x) \neq 0$ we get from (3.39), (3.40),

$$A_0^{(x)}(t, x) - K(t, x)A_0(t, x) = f(t, x), \quad D_0^{(x)}(t, x) - K(t, x)D_0(t, x) = g(t, x), \tag{3.41}$$

where

$$K(t, x) = \frac{S(t, x)T(t, x) - (ST)^\sigma(t, x)}{\nu(x)(1 - (ST)^\sigma(t, x))}, \tag{3.42}$$

$$f(t, x) = \frac{1 - ST}{\nu(x)(1 - (ST)^\sigma)}\left(A_1[Q^\rho T - T^{\sigma\rho^{-1}} Q^\sigma + RS - (RS)^\sigma] + \frac{(1 - ST)^{(t)}}{1 - ST}\right), \tag{3.43}$$

$$g(t, x) = \frac{1 - ST}{\nu(1 - (ST)^\sigma)}\left(D_{-1}[R^\rho S - S^{\sigma\rho^{-1}} R^\sigma + QT - (QT)^\sigma] + \frac{(1 - ST)^{(t)}}{1 - ST}\right). \tag{3.44}$$

Solving (3.41) by a variation of parameters formula (see Ref. 6) we get

$$A_0(t, x) = c_1 e_K(t, x, x_0) + \int_{x_0}^{x} e_K(t, x, \rho(y)) f(t, y)\nabla y, \tag{3.45}$$

$$D_0(t, x) = c_2 e_K(t, x, x_0) + \int_{x_0}^{x} e_K(t, x, \rho(y)) g(t, y)\nabla y, \tag{3.46}$$

where c_1, c_2 are the constants, $e_K(t, x, y)$ is the exponential function on a space scale (see (2.23)).

In the case $\nu(x) = 0$, $\mu(t) \neq 0$ we have $A_0^\rho = A_0$, $D_0^\rho = D_0$, and from (3.39), (3.40) we get

$$A_0(t,x) = \frac{1-ST}{(ST)^\sigma - ST}\left[A_1\left(QT-(QT)^\sigma + RS - (RS)^\sigma\right) + \frac{(1-ST)^{(t)}}{1-ST}\right], \tag{3.47}$$

$$D_0(t,x) = \frac{1-ST}{(ST)^\sigma - ST}\left[D_{-1}\left(RS-(SR)^\sigma + QT - (QT)^\sigma\right) + \frac{(1-ST)^{(t)}}{1-ST}\right]. \tag{3.48}$$

Combining (3.45), (3.47) and (3.46), (3.48) we get (2.21), (2.22).

In the case $\nu(x) \equiv \mu(t) \equiv 0$ conditions (3.39), (3.40) are satisfied for any $A_0(t, x)$, $D_0(t, x)$ if

$$(S(t,x)T(t,x))^{(t)} = 0, \tag{3.49}$$

that is $S(t, x)T(t, x)$ does not depend on time.

Remark 3.1. Using $v_1^\rho(t,x) = v_1(t,x) - \nu(x)v_1^{(x)}(t,x)$ one may rewrite the spectral problem (2.11) in terms of space nabla derivatives. Indeed

$$(1-ST)(v_1(t,x) - \nu(x)v_1^{(x)}(t,x)) = (RS+\Lambda_1)v_1(t,x) + (Q+\Lambda_2 S)\,v_2(t,x),$$

$$(1-ST)(v_2(t,x) - \nu(x)v_2^{(x)}(t,x)) = (R+T\Lambda_1)v_1(t,x) + (QT+\Lambda_2)\,v_2(t,x),$$

or

$$(1-ST)\nu(x)v_1^{(x)}(t,x) = (1-ST-RS-\Lambda_1)v_1(t,x) - (Q+\Lambda_2 S)\,v_2(t,x),$$

$$(1-ST)\nu(x)v_2^{(x)}(t,x) = -(R+T\Lambda_1)v_1(t,x) + (1-ST-QT-\Lambda_2)v_2(t,x),$$

or in the matrix form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^{(x)} = \frac{1}{\nu(x)(1-ST)}\begin{pmatrix} 1-ST-RS-\Lambda_1 & -Q-\Lambda_2 S \\ -R-T\Lambda_1 & 1-ST-QT-\Lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{3.50}$$

Let

$$v(t,x) = \begin{pmatrix} v_1(t,x) \\ v_2(t,x) \end{pmatrix}, \quad w(t,x) = \begin{pmatrix} w_1(t,x) \\ w_2(t,x) \end{pmatrix}$$

be a solution of (2.11) and define the Wronskian by

$$W(t,v,w) = v_1(t,x)w_2(t,x) - w_1(t,x)v_2(t,x). \tag{3.51}$$

One can prove that

$$W^\rho(t,v,w) = det(M)W(t,v,w) = \frac{1-R(t,x)Q(t,x)}{1-S(t,x)T(t,x)}W(t,v,w). \tag{3.52}$$

Indeed, using the notation

$$m_{11} = \frac{RS+\Lambda_1}{1-ST}, \quad m_{12} = \frac{Q+\Lambda_2 S}{1-ST}, \quad m_{21} = \frac{R+T\Lambda_1}{1-ST}, \quad m_{22} = \frac{QT+\Lambda_2}{1-ST}$$

from (2.11) we get

$$v_1 w_2^\rho - w_1 v_2^\rho = m_{22}(v_1 w_2 - w_1 v_2), \quad v_2 w_2^\rho - w_2 v_2^\rho = m_{21}(v_2 w_1 - w_2 v_1)$$

$$W^\rho(t,v,w) = v_1^\rho w_2^\rho - w_1^\rho v_2^\rho = m_{11}(v_1 w_2^\rho - w_1 v_2^\rho) + m_{12}(v_2 w_2^\rho - w_2 v_2^\rho)$$

$$= (m_{11}m_{22} - m_{12}m_{21})(v_1 w_2 - w_1 v_2) = \frac{\Lambda_1\Lambda_2 - RQ}{1-ST} = \frac{1-RQ}{1-ST}.$$

Further in view of (2.14) we get (3.52) since $det(M) = m_{11}m_{22} - m_{12}m_{21}$.

Assuming that the all functions Q, R, S, T vanish sufficiently rapidly as $|x| \to \infty$ from (3.24), (3.25), (3.28) we get

$$C^{\pm} = \lim_{x\to\pm\infty}(C_0 + C_1 z) = \lim_{x\to\pm\infty}(RD_{-1} + A_1^{\rho}T^{\sigma}z) = 0,$$
$$B^{\pm} = \lim_{x\to\pm\infty}(B_0 + B_{-1}z^{-1}) = \lim_{x\to\pm\infty}(QA_1 + D_{-1}S^{\sigma\rho^{-1}}z^{-1}) = 0. \quad (3.53)$$

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