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ASYMPTOTIC BEHAVIOR OF A PLANAR DYNAMIC SYSTEM

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ABSTRACT. We investigate the asymptotic solutions of the planar dynamic systems and the second order equations on a time scale by using a new version of Levinson's asymptotic theorem. In this version the error estimate is given in terms of the characteristic (Riccati) functions which are constructed from the phase functions of an asymptotic solution. It means that the improvement of the approximation depends essentially on the asymptotic behavior of the Riccati functions. We describe many different approximations using the flexibility of this approach. As an application we derive the analogue of D'Alembert's formula for the one dimensional wave equation in a discrete time.

1. Introduction. Consider the planar dynamic system

$$\varphi^{\Delta}(t) = A(t)\varphi(t), \quad A(t) = \begin{pmatrix} a_{11}(t) & a_{11}(t) \\ a_{11}(t) & a_{11}(t) \end{pmatrix}, \quad t > t_0 \tag{1.1}$$

on a time scale $\mathbf{T}$ (an arbitrary nonempty closed subset of the real numbers). Here $\varphi^{\Delta}(t)$ is the delta derivative on a time scale ([**3, 17**]).

The theory of asymptotic integration of systems of differential equations was developed in [**10, 12, 16, 23, 26**]. In 1948, Levinson ([**23**]) discovered a powerful and simple method of finding asymptotic solutions. This method is based on splitting matrix $A(t)$ into a sum of diagonal and perturbation matrices. Levinson proved that, if the L_1-norm of perturbation matrix is bounded, then the diagonal matrix gives the asymptotic solution of the system. Levinson's result was further developed in [**12, 14, 15**] by using diagonalization. This theory was extended for difference equations in [**2, 3, 11, 25**]. The theory of asymptotic solutions (based on Levinson's asymptotic theo-

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rem) of dynamic systems on a time scale had been developed in papers [**5, 4, 6, 9, 13, 24**].

In [**18, 20**], generalizing Levinson's method, the author suggested splitting the matrix $A(t)$ into two matrices: $A(t) = S(t) + P(t)$, where $S(t) = \Phi^{\Delta}(t)\Phi^{-1}(t)$ is the solvable matrix ($\Phi(t)$ is an unknown approximate fundamental matrix that is constructed from some phase functions θ_1, θ_2), and $P(t)$ is a perturbation matrix. The key new formula (see (5.8) below) in our approach is that the perturbation $P(t)$ could be written in terms of the characteristic (Riccati) functions (see (2.1) below). Using this formula, we show that the error of the approximation is small if the weighted L_1-norm of the Riccati functions is bounded (see [**18, 20**] or Theorem 2.2 below). This estimate shows that the improvement of the approximation depends essentially on the behavior of the Riccati functions. The advantage of our approach in comparison to Levinson's theorem is that we obtain asymptotic representations not only for the solutions (position functions in physics applications), but for the derivatives (velocities) as well, and we don't use the diagonal structure of the fundamental matrix $\Phi(t)$ or diagonalization at all. Since the formulas for the n-dimensional non-autonomous systems are complicated, we consider only the planar dynamic system, which includes important second order non-autonomous discrete and continuous dynamic equation. Consideration of more involved n-dimensional non-autonomous systems may be a topic for future study. Note that it could be shown (see [**21**]) that the error of asymptotic solutions of three dimensional non-autonomous systems depends on characteristic (Weierstrass) functions, which are very hard to study since they are highly non-linear.

In [**18, 19, 20**] the Riccati function approach was used for the study of stability. In [**22**], we applied this method to study asymptotic solutions of non-autonomous Dirac system. Note that the Riccati function approach is essential in oscillation theory as well (see [**21**]).

Example 1.1. Consider the Cauchy problem for the one dimensional difference-differential wave equation

$$u_n^{\Delta\Delta}(x) = (a^2 + b_n)u_{nxx}(x) + c_n u_{nx}(x), \quad n = 1, 2, \dots, \tag{1.2}$$

$$u_1(x) = f(x), \quad u_2(x) = u_1(x), \tag{1.3}$$

for $x \in R_x$, where $\Delta\Delta$ is the second forward difference operator by the time variable $t = n$, R_x is the set of real numbers, $u_{nx}(x)$ is the first derivative of $u_n(x)$ by x variable, $u_{nxx}(x)$ is the second derivative by x, a is a positive number (speed of the wave), b_n, c_n are given sequences, $f(x)$ is a tempered distribution, that is, $f(x) \in S'(R_x)$ and $S(R_x)$ is Schwartz's space.

Using Fourier transformation by x,

$$F[u_n(x)] = \widehat{u}_n(\xi) = \int_{-\infty}^{\infty} e^{ix\xi} u_n(x)\, dx, \tag{1.4}$$

we get from (1.2) the difference dynamic equation

$$\begin{gathered} \widehat{u}_n^{\Delta\Delta}(\xi) + (\xi^2 a^2 + \xi^2 b_n + i\xi c_n)\widehat{u}_n(\xi) = 0, \\ \widehat{u}_1(\xi) = \widehat{f}(\xi), \qquad \widehat{u_2}(\xi) = \widehat{u}_1(\xi). \end{gathered} \tag{1.5}$$

It is known that Fourier transformation and its inverse map $S'(R_x)$ onto $S'(R_x)$, and we are assuming $u_n \in S'(R_x)$. If

$$\sum_{k=1}^{\infty} |b_k| < \infty, \qquad \sum_{k=1}^{\infty} |c_k| < \infty, \tag{1.6}$$

then solutions of (1.5) have asymptotic representations

(1.7)
$$\widehat{u}(t,\xi) = [C_1(\xi) + \varepsilon_{1n}(\xi)] \sin_{a\xi}(n) + [C_2(\xi) + \varepsilon_{2n}(\xi)] \cos_{a\xi}(n),$$
(1.8)
$$\widehat{u}_n^{\Delta}(\xi) = a\xi[C_{1n}(\xi) + \varepsilon_{1n}(\xi)] \cos_{a\xi}(n) + a\xi[C_{2n}(\xi) + \varepsilon_{2n}(\xi)] \sin_{a\xi}(n),$$

$$\lim_{n\to\infty} \varepsilon_{jn}(\xi) = 0, \quad \text{uniformly for all} \quad \xi \in R_\xi, \quad j = 1, 2, \dots, \tag{1.9}$$

(see Theorem 3.3 below). Here $C_{1,2} \in S'(R_x)$ are arbitrary functions, $\sin_{a\xi}(n)$ and $\cos_{a\xi}(n)$ are trigonometric functions (see **[7]**) on a discrete time scale $\mathbf{Z}$.

Using inverse Fourier transformation, one can derive from (1.7) the discrete time analogue of D'Alembert's formula:

$$\begin{gathered} u_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} [\widehat{f}(\xi) + \varepsilon_{2n}(\xi)] \cos_{a\xi}(n)\, d\xi, \\ \lim_{n\to\infty} \varepsilon_{2n}(\xi) = 0. \end{gathered} \tag{1.10}$$

Similar asymptotic representations may be written for the multidimensional wave equation. The usefulness of such asymptotic representations for the study of the behavior of solutions of dynamic equations of mathematical physics (**[22]**) is well known, but there are no results for non-autonomous multidimensional difference-differential equations.

If condition (1.6) fails, then asymptotic behavior of solutions of (1.2) may be totally different, which is why in this paper we discuss many different asymptotic solutions.

This example shows that, using the error estimate described here, one can obtain an asymptotic representation for solutions of multidimensional equations of mathematical physics for discrete or continuous time.

In this paper we will show how to find different asymptotic representations of solutions of (1.1) by using the Riccati function approach. Note that some asymptotic approximations described here are new even for differential equations (see, for example, Theorem 2.4 and Theorem 3.5 below).

Recall some basic definitions from the theory of time scales **[7, 17]**.

If the time scale $\mathbf{T}$ has a left-scattered minimum m, then $\mathbf{T}^k = \mathbf{T} - \{m\}$; otherwise, $\mathbf{T}^k = \mathbf{T}$. Here we consider the time scales with $t \geq t_0$, and $\sup \mathbf{T} = \infty$.

For $t \in \mathbf{T}$, we define the forward jump operator

$$\sigma(t) = \inf\{s \in \mathbf{T}, s > t\}. \tag{1.11}$$

The forward graininess function $\mu : \mathbf{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{1.12}$$

If $\sigma(t) > t$, we say that t is right-scattered. If $t < \infty$ and $\sigma(t) = t$, then t is called right-dense.

For $f : \mathbf{T} \to \mathbf{R}$ and $t \in \mathbf{T}^k$ define the delta (see **[2]**) derivative $f^{\Delta}(t)$ to be the number (provided it exists) with the property that, for given any $\epsilon > 0$, there exist a $\delta > 0$ and a neighborhood $U = (t-\delta, t+\delta) \cap \mathbf{T}$

of t such that

$$|f^{\sigma}(t) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U, \tag{1.13}$$

$$f^{\sigma}(t) \equiv f(\sigma(t)). \tag{1.14}$$

The set of rd-continuous functions in $\mathbf{T}_0 = \mathbf{T} \bigcap (t_0, \infty)$ is denoted by C_{rd}. Let C^1_{rd} be a set of delta differentiable functions in $\mathbf{T}_0$ such that their delta derivatives are C_{rd} functions. We assume that $A(t)$ is 2×2 matrix function from C^1_{rd}. We say that a function $f : \mathbf{T}_0 \to R$ is regressive if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbf{T}_0$.

2. Asymptotic solutions of the dynamic systems in terms of the phase functions. Introduce characteristic (Riccati) function of system (1.1)

(2.1)

$$\begin{aligned} CA_k(t) &= CA(\theta_k) \\ &= \frac{a^{\sigma}_{12}}{a_{12}}\left(\theta^2_k - \theta_k Tr(A) + \det(A) + a_{12}(1 + \mu\theta_k)\left(\frac{\theta_k - a_{11}}{a_{12}}\right)^{\Delta}\right), \end{aligned}$$

and auxiliary function

$$HA(t) = \frac{CA_1(t) - CA_2(t)}{\theta_1 - \theta_2}, \tag{2.2}$$

or

$$\begin{aligned} HA(t) &= \frac{\theta^{\sigma} + \theta}{\theta}\theta_1 \\ &\quad + \frac{\theta^{\Delta}}{\theta} - 2\theta^{\sigma} - Tr(A) - \frac{a^{\Delta}_{12}}{a_{12}}(1 + \mu a_{22}) + \mu(\theta^{\Delta}_1 - a^{\Delta}_{11}), \end{aligned} \tag{2.3}$$

where we suppressed the time variable t, and $\sigma = \sigma(t)$ is the forward jump operator,

$$\theta(t) \equiv \frac{\theta_1(t) - \theta_2(t)}{2} \tag{2.4}$$

(2.5)

$$\mathrm{Tr}\,(A(t)) \equiv a_{11}(t) + a_{22}(t), \quad \det(A(t)) \equiv a_{11}(t)a_{22}(t) - a_{12}(t)a_{21}(t),$$

$$W[a_{11}, a_{12}] \equiv a_{11}(t)a_{12}^{\Delta}(t) - a_{11}^{\Delta}(t)a_{12}(t). \tag{2.6}$$

Asymptotic behavior of solutions of system (1.1) could be described in terms of phase and characteristic functions by using representation of the fundamental matrix $\Phi(t)$

$$\begin{aligned} \Phi(t) &= \begin{pmatrix} 1 & 1 \\ U_1(t) & U_2(t) \end{pmatrix} \begin{pmatrix} e_1(t) & 0 \\ 0 & e_2(t) \end{pmatrix}, \\ U_j(t) &= \frac{\theta_j(t) - a_{11}(t)}{a_{12}(t)}, \quad j = 1, 2, \end{aligned} \tag{2.7}$$

where
(2.8)

$$e_j(t) = e_{\theta_j}(t, t_0) = \exp \int_{t_0}^{t} \lim_{m \searrow \mu(s)} \frac{\operatorname{Log}(1 + m\theta_j(s))\Delta s}{m}, \quad j = 1, 2, \ldots,$$

are exponential functions on a time scale ([**7, 17**]). Phase functions θ_j, $j = 1, 2, \ldots$, may be found as asymptotic solutions of characteristic equations $CA_j(t) = CA(\theta_j) = 0$.

Note that the diagonal fundamental matrix

$$\begin{pmatrix} e_1(t) & 0 \\ 0 & e_2(t) \end{pmatrix}$$

is used in Levinson's asymptotic theorem.

We introduce the 2×2 matrix function

$$K(t) = (\Phi^{\sigma}(t))^{-1}(A(t)\Phi(t) - \Phi^{\Delta}(t)), \tag{2.9}$$

with Euclidean norm

$$\|K(t)\| = \sqrt{\sum_{i,j=1}^{2} |K_{ij}(t)|^2}. \tag{2.10}$$

Theorem 2.1 ([**19**]). *Assume there exists a matrix-function $\Phi \in C_{rd}^1$ such that $\Phi^{\sigma}(t)$ is invertible and $\|K(t)\| \in C_{rd}$. Then every solution of* (1.1) *may be represented in the form*

$$\varphi(t) = \Phi(t)[C + \varepsilon(t)], \tag{2.11}$$

where

$$\|\varepsilon(t)\| \leq \|C\|\left(-1+\exp\int_t^\infty \|K(s)\|\Delta s\right). \tag{2.12}$$

Using the structure of fundamental matrix (2.7), one can prove the following theorem.

Theorem 2.2. **[20]** *Assume* $a_{12}(t) \neq 0$, $A \in C^1_{rd}$ *and there exist regressive phase functions* $\theta_{1,2} \in C^1_{rd}$ *such that* $\Phi^\sigma(t)$ *is invertible, that is,*

$$\left|\frac{\theta^\sigma(t)e_1^\sigma(t)e_2^\sigma(t)}{a_{12}^\sigma(t)}\right| \neq 0, \tag{2.13}$$
$$\theta(t) = \frac{\theta_1(t)-\theta_2(t)}{2}, \quad t > t_0,$$

and

$$M(t) \in C_{rd}, \qquad \int_t^\infty M(s)\Delta s < \infty, \quad t > t_0, \tag{2.14}$$

where

$$M(t) = \max_{k,j=1,2}\left|\frac{a_{12}(\sigma(t))e_j(t)CA_j(t)}{a_{12}(t)\theta(\sigma(t))e_k(\sigma(t))}\right|. \tag{2.15}$$

Then every solution of (1.1) *may be represented in the form* (2.7), (2.11), *where*

$$\|\varepsilon(t)\| \leq \|C\|\left(-1+\exp\int_t^\infty cM(s)\Delta s\right). \tag{2.16}$$

To apply Theorem 2.2, one needs to find out the asymptotic fundamental matrix Φ, with phase functions $\theta_{1,2}$. To show how to do that consider the simple second order equation:

$$\psi^{\Delta\Delta}(t) + Q(t)\psi(t) = 0, \quad t \in \mathbf{T},\ t \geq t_0 > 0. \tag{2.17}$$

To find phase functions, one can solve approximately the characteristic equation (see (3.5) below) of equation (2.17):

$$CL(\theta_1) = \theta_1^\sigma(t)\theta_1(t) + \theta_1^\Delta(t) + Q(t) = 0$$

for unknown phase functions $\theta_{1,2}$ considering the following basic cases.

1. Bernoulli's approximation: $Q(t)$ is small when $t \to \infty$,

$$CL(\theta_1) = \theta_1^\sigma(t)\theta_1(t) + \theta_1^\Delta(t) + Q(t) \approx \theta_1^\sigma\theta_1 + \theta_1^\Delta = 0, \qquad \theta_1(t) = \frac{1}{t}.$$

2. Trigonometric function approximation: when $t \to \infty$, $Q(t) \to m^2$, m is a positive number,

$$CL(\theta_1) \approx \theta_1^2(t) + m^2 = 0, \quad \theta_1 = im,\ \theta_2 = -im.$$

3. Linear equation approximation: $\theta_1^\sigma(t)\theta_1(t)$ is small when $t \to \infty$,

$$CL(\theta_1) \approx \theta_1^\Delta(t) + Q(t) = 0, \qquad \theta_1(t) = -\int_{t_0}^t Q(s)\Delta s.$$

4. Eigenvalue first approximation: $\theta_1^\Delta(1+\mu\theta_1)$ is small when $t \to \infty$,

$$CL(\theta_1) \approx \theta_1^2(t) + Q(t) = 0, \quad \theta_1(t) = i\sqrt{Q(t)},\ \theta_2(t) = -i\sqrt{Q(t)}.$$

5. Eigenvalue second (JWKB) approximation:

$$\theta_{1,2}(t) = \pm i\sqrt{Q(t)} - \frac{Q'(t)}{4Q(t)}.$$

6. Eigenvalue third (Hartman-Wintner) approximation: $\theta_1(t) = iA(t)\sqrt{Q(t)}$, A^Δ is small when $t \to \infty$.

By using Bernoulli's approximation phase functions:

$$\theta_1(t) = -P_1(t), \qquad \theta_2(t) = \frac{1}{\int_{t_0}^t \frac{\Delta s}{1-P_1(s)\mu(s)}} - P_1(t), \tag{2.18}$$

where

$$P_1^\sigma(t) + P_1(t) = -Tr(A(t)) - \mu(t)a_{11}^\Delta(t) - \frac{a_{12}^\Delta(t)}{a_{12}(t)}(1 + \mu(t)a_{22}(t)), \tag{2.19}$$

$$\begin{aligned} R(t) &= Q(t) - P_1^2(t) - P_1^\Delta(t), \\ Q(t) &= \frac{a_{12}^\sigma \det(A(t))}{a_{12}(t)} - a_{12}^\sigma(t)\left(\frac{a_{11}(t)}{a_{12}(t)}\right)^\Delta, \end{aligned} \tag{2.20}$$

from Theorem 2.2, we deduce the following theorem.

Theorem 2.3. *Assume* $a_{12}(t) \neq 0$, $A(t) \in C^1_{rd}$, $a_{12}(t) \in C^2_{rd}$, *phase functions* $\theta_{1,2}$ *given by* (2.18) *are regressive, conditions* (2.13) *and* (2.14) *are satisfied, where*

$$M(t) = \max_{j=1,2} \left| \frac{e_q^{\pm 1}(t) a_{12}(\sigma(t)) R(t)}{a_{12}(t)(1+\mu(t)\theta_j(t))} \int_{t_0}^{\sigma(t)} \frac{\Delta s}{1 - P_1(s)\mu(s)} \right|, \tag{2.21}$$

$$q(t) = \frac{\theta_2(t) - \theta_1(t)}{1 - P_1(t)\mu(t)}. \tag{2.22}$$

Then every solution of (1.1) *may be represented in the form* (2.7), (2.11), (2.18), *with the error estimate* (2.16).

Example 2.1. From Theorem 2.3, it follows that if

$$\int_{t_0}^{\infty} t\sigma(t)|Q(t)|\Delta t < \infty, \tag{2.23}$$

then solutions of dynamic equation (2.17) may be written in the form

$$u(t) = [C_1 + \varepsilon_1(t)]\frac{t - t_0}{t_0} + C_2 + \varepsilon_2(t), \qquad u^{\Delta}(t) = \frac{C_1 + \varepsilon_1(t)}{t_0}, \tag{2.24}$$

where

$$\lim_{t\to\infty} \varepsilon_j(t) = 0, \quad j = 1, 2, \dots . \tag{2.25}$$

For the discrete case ($\mathbf{T} = \mathbf{Z}$), from the same theorem it follows that if

$$\sum_{k=1}^{\infty} k(k+1)Q_k < \infty, \tag{2.26}$$

then solutions of the forward second difference equation

$$\psi_n^{\Delta\Delta} + Q_n \psi_n = 0, \quad n \geq 1, \tag{2.27}$$

have asymptotic representation

$$\psi_n = (C_1 + \varepsilon_{1n})n + C_2 + \varepsilon_{2n}, \qquad \psi_n^{\Delta} = C_1 + \varepsilon_{1n}, \quad n \geq 1, \tag{2.28}$$

where

$$\lim_{n\to\infty} \varepsilon_{jn} = 0, \quad j = 1, 2. \tag{2.29}$$

Note that condition (2.23) is stronger than the well-known condition (see **[9]**)

$$\int_{t_0}^{\infty} \sigma(t)|Q(t)|\Delta t < \infty \tag{2.30}$$

of asymptotic representation of solutions of (2.17) in the form

$$\psi(t) = C_1 t + C_2 + o(1), \quad t \to \infty,$$

but under condition (2.23) we get asymptotic representation for the first derivatives as well.

By using the linear equation approximation phase functions

$$\begin{aligned} \theta_1(t) &= -\int_{t_0}^{t} \frac{R(s)\Delta s}{1-\mu(s)P_1(s)} - P_1(t), \\ \theta_2(t) &= \theta_1(t) + \frac{1}{\int_{t_0}^{t} \frac{e_\Lambda(t,\sigma(s)\Delta s}{1-\mu(s)[P_1^\sigma(s)+P_1(s)+\theta_1^\sigma(s)]}}, \end{aligned} \tag{2.31}$$

$$\Lambda(t) = \frac{\theta_1(t) + \theta_1^\sigma(t) + P_1^\sigma(t) + P_1(t)}{1-\mu(t)[P_1^\sigma(t) + P_1(t) + \theta_1^\sigma(t)]}, \tag{2.32}$$

from Theorem 2.2 we deduce the following theorem.

Theorem 2.4. *Assume $a_{12}(t) \neq 0$, $A(t) \in C_{rd}^1$, $a_{12}(t) \in C_{rd}^2$, the phase functions $\theta_{1,2}$ given by* (2.31) *are regressive, and conditions*

$$\begin{aligned} &1 - \mu(t)P_1(t) \neq 0, \\ &\left| \frac{e_1(t)e_2(t)}{a_{12}(t)\int_{t_0}^{t} \frac{e_\Lambda(t,\sigma(s)\Delta s}{1-\mu(s)[P_1^\sigma(s)+P_1(s)+\theta_1^\sigma(s)]}} \right| \neq 0, \quad t > t_0, \end{aligned} \tag{2.33}$$

(2.14) *are satisfied, where*

$$\begin{aligned} M(t) = \max &\left| \frac{a_{12}^\sigma(t)e_q^{\pm 1}(t)}{2\theta(t)a_{12}(t)[1+\mu(t)\theta_j(t)]} \right. \\ &\times \left. \int_{t_0}^{t} \frac{R(s)\Delta s}{1-\mu(s)P_1(s)} \int_{\sigma_0}^{\sigma} \frac{R(s)\Delta s}{1-\mu(s)P_1(s)} \right|, \end{aligned} \tag{2.34}$$

(2.35)

$$q(t)=\frac{\theta_2-\theta_1}{1+\theta_1\mu}$$
$$=\frac{1}{\Big(1-\mu(t)P_1(t)-\mu(t)\int_{t_0}^t\frac{R(s)\Delta s}{1-\mu(s)P_1(s)}\Big)\int_{t_0}^t\frac{e_\Lambda(t,\sigma(s)\Delta s}{1-\mu(s)[P_1^\sigma(s)+P_1(s)+\theta_1^\sigma(s)]}}.$$

Then every solution of (1.1) *may be represented in the form* (2.7), (2.11) (2.31), *with the error estimate* (2.16).

Example 2.2. For equation (2.17) with

$$Q(t)=t^\alpha,\qquad \mu(t)=0,\qquad -3<\alpha<-2.5,$$

condition (2.14) is satisfied, and Theorem 2.4 is applicable, but Theorem 2.3 is not, since (2.23) fails.

Remark 2.1. If, for a given function $P(t)$, we define

$$P_1(t)=\begin{cases}\int_{t_0}^t e_{-2/\mu}(t,s)\frac{2P(s)\Delta s}{\mu(s)}, & \mu(t)\neq 0,\\ P(t), & \mu(t)\equiv 0,\end{cases}\tag{2.36}$$

then

$$P_1^\sigma(t)+P_1(t)=2P(t).\tag{2.37}$$

3. Asymptotic solutions of the second order dynamic equations. Consider the second order equation on a time scale

$$L[\psi]=\psi^{\Delta\Delta}(t)+(P_1^\sigma(t)+P_1(t))\psi^\Delta(t)+Q(t)\psi(t)=0,\quad t>t_0,\tag{3.1}$$

with complex valued coefficients $P_1(t)$ and $Q(t)$. Denote

$$R(t)=Q(t)-P_1^2(t)-P_1^\Delta(t).\tag{3.2}$$

Note that, in continuous time scale $\mathbf{T}=R$, the function $R(t)$ is invariant of the transformation $\psi(t)\to u(t)w(t)$.

The functions $u_j(t)$ are called asymptotic solutions of (3.1) if the solutions $\psi(t)$ of (3.1) may be represented in the form

$$\begin{aligned}\psi(t)&=(C_1+\varepsilon_1(t))u_1(t)+(C_2+\varepsilon_2(t))u_1(t),\\ \psi^\Delta(t)&=(C_1+\varepsilon_1(t))u_1^\Delta(t)+(C_2+\varepsilon_2(t))u_1^\Delta(t),\end{aligned}\tag{3.3}$$

where

$$\lim_{t\to\infty} \varepsilon_j(t) = 0, \quad j = 1, 2, \dots . \tag{3.4}$$

Define the characteristic functions $CL_j(t) = CL(\theta_j)$, $j = 1, 2, \dots$, of equation (3.1), and the auxiliary function $HL(t)$

$$CL_j(t) = \frac{L[e_{\theta_j}(t, t_0)]}{e_{\theta_j}(t, t_0)}$$
$$= \theta_j^{\Delta}(t) + \theta_j^{\sigma}(t)\theta_j(t) + (P_1^{\sigma}(t) + P_1(t))\theta_j(t) + Q(t),$$

$$HL(t) = \frac{CL_1(t) - CL_2(t)}{\theta_1(t) - \theta_2(t)}. \tag{3.5}$$

Theorem 3.1. *Assume $P_1, Q \in C_{rd}$, and there exist regressive complex-valued functions $\theta_{1,2} \in C_{rd}^1$ such that the conditions*

$$|\theta(t)e_1(t, t_0)e_2(t, t_0)| \neq 0, \qquad \theta(t) \equiv \frac{\theta_1(t) - \theta_2(t)}{2}, \quad t > t_0, \tag{3.6}$$

$$M(t) \in C_{rd}, \qquad \int_t^{\infty} M(s)\Delta s < \infty, \quad t > t_0, \tag{3.7}$$

are satisfied with

$$M(t) = \max_{k,j=1,2} \left| \frac{e_j(t) CL_j(t)}{2\theta^{\sigma}(t)(1 + \mu(t)\theta_k(t))e_k(t)} \right|. \tag{3.8}$$

Then every solution of equation (3.1) *may be represented in form* (3.3), (3.4)*, where*

$$u_k(t) = e_k(t) = e_{\theta_k}(t, t_0), \quad k = 1, 2, \tag{3.9}$$

and error estimate (2.16) *is true with the function $M(s)$ given by* (3.8).

Using Bernoulli's approximation, we deduce from Theorem 3.1 the following theorem.

Theorem 3.2. *Assume $P_1 \in C_{rd}^1$, $Q \in C_{rd}$, functions*

$$\left(\int_{t_0}^{t} \frac{\Delta s}{1 - P_1 \mu(s)} \right)^{-1} - P_1(t),$$

$-P_1(t)$ *are regressive, and conditions* (3.6) *and* (3.7) *are satisfied with*

$$\begin{aligned}
&\theta_1(t) = -P_1(t),\\
(3.10)\qquad &\theta_2(t) = \frac{1}{\int_{t_0}^{t} \frac{\Delta s}{1-P_1(s)\mu(s)}} - P_1(t),\\
&M(t) = \max_{k=1,2} \left| \frac{e_q^{\pm 1}(t) R(t) \int_{t_0}^{\sigma} \frac{\Delta s}{1-P_1(s)\mu(s)}}{1+\mu(t)\theta_k(t)} \right|,\\
(3.11)\qquad &q(t) = \frac{\left(1-P_1(t)\mu(t)\right)^{-1}}{\int_{t_0}^{t} \frac{\Delta s}{1-P_1(s)\mu(s)}},
\end{aligned}$$

and $R(t)$ *is defined in* (3.2)*. Then every solution of the equation* (3.1) *may be represented in the form* (3.3)*,* (3.9)*,* (3.10)*, and error estimate* (2.16) *is satisfied with the function* $M(s)$ *given by* (3.11)*.*

Using the trigonometric function approximation, we deduce from Theorem 3.1 the following theorem.

Theorem 3.3. *Assume* $P_1 \in C^1_{rd}$*,* $Q \in C_{rd}$*,* $-P_1(t)$ *is regressive, and conditions* (3.6) *and* (3.7) *are satisfied, where*

$$(3.12)\qquad \theta_1(t) = im - P_1(t), \qquad \theta_2(t) = -im - P_1(t),$$

$$\begin{aligned}
&M(t) = \max_{j=1,2} \left| \frac{e_q^{\pm 1}(t)(R(t)-m^2)}{m(1+\mu(t)\theta_j(t))} \right|,\\
(3.13)\qquad &q(t) = \frac{-2im}{1+im\mu(t)-P_1(t)\mu(t)}, \quad j = 1, 2, \dots .
\end{aligned}$$

Then every solution of (3.1) *may be represented in the form* (3.3)*,* (3.9)*,* (3.12)*, and error estimate* (2.16) *is true with the function* $M(t)$ *given by* (3.13)*.*

Using the eigenvalue first approximation, we deduce from Theorem 3.1 the following theorem.

Theorem 3.4. *Assume $P_1 \in C^2_{rd}$, $Q \in C^1_{rd}$, conditions* (3.6) *and* (3.7) *are satisfied, where*

$$\theta_1(t) = i\sqrt{R(t)} - P_1(t),$$

$$\theta_2 = -i\sqrt{R(t)} - P_1(t) \tag{3.14}$$

$$q(t) = \frac{-2i\sqrt{R(t)}}{1+\mu(t)\theta_1(t)},$$

$$M(t) = \max\left|\frac{e_q^{\pm 1}(t)}{2\mu(t)(1+\mu(t)\theta_j(t))}\left(1-\sqrt{\frac{R(t)}{R^\sigma(t)}}\right)\left(1 \pm i\mu(t)\sqrt{R(t)} - \mu(t)P_1(t)\right)\right|, \tag{3.15}$$

and functions θ_1 and θ_2 are regressive. Then every solution of (3.1) *may be represented in form* (3.3), (3.9), (3.14) *and error estimate* (2.16) *is satisfied with the function $M(s)$ given by* (3.15).

Using linear equation approximation, we deduce from Theorem 3.1 the following theorem.

Theorem 3.5. *Assume $P_1 \in C^1_{rd}$, $Q \in C_{rd}$, and conditions* (3.6), (3.7) *are satisfied, where*

$$\theta_1(t) = -\int_{t_0}^{t} \frac{R(s)\Delta s}{1-P_1(s)\mu(s)} - P_1(t), \tag{3.16}$$

$$\theta_2(t) = \theta_1(t) + \frac{1}{\int_{t_0}^{t} \frac{e_\Lambda(t,\sigma(s))\Delta s}{1-\mu(s)[\theta_1^\sigma(s)+2P(s)]}},$$

$$M(t) = \left|\frac{e_q^{\pm 1}(t)}{2\theta(t)(1+\mu(t)\theta_j(t))} \times \int_{\sigma_0}^{\sigma} \frac{R(s))\Delta s}{1-P_1(s)\mu(s)} \int_{t_0}^{t} \frac{R(s))\Delta s}{1-P_1(s)\mu(s)}\right|, \tag{3.17}$$

$$q(t) = \frac{\theta_2(t)-\theta_1(t)}{1+\mu(t)\theta_1(t)}, \qquad \Lambda(t) = \frac{\theta_1(t)+\theta_1^\sigma(t)+2P(t)}{1-\mu(t)\theta_1^\sigma(t)-2P(t)\mu(t)}, \tag{3.18}$$

and functions θ_1 and θ_2 are regressive. Then every solution of (3.1) *may* *be represented in the form* (3.3), (3.9), (3.16), *and the error estimate* (2.16) *is true with the function* $M(s)$ *given by* (3.17).

Using the Hartman-Wintner approximation, we deduce from Theorem 3.1 the following theorem.

Theorem 3.6. *Assume* $P_1 \in C^3_{rd}$, $Q \in C^2_{rd}$, *conditions* (3.6) *and* (3.7) *are satisfied, where*

$$\theta_{1,2}(t) = \pm i\sqrt{R(t)}\left(\sqrt{G^2(t)-p^2(t)} - p(t)\right) - P_1(t), \tag{3.19}$$

$$G(t) = \left(\frac{R(t)}{R^\sigma(t)}\right)^{1/4},$$

$$M(t) = \max_{k,j=1,2}\left|\frac{e_q^{\pm 1} A_j^\Delta (1 - P_1\mu + \mu A_j\sqrt{R})}{2(1+\mu\theta_k)\sqrt{(G^2-p^2)^\sigma}}\right|, \tag{3.20}$$

$$p = \frac{(1-P_1\mu)G^4 R^\Delta}{2R^{3/2}(1+G^2)}, \qquad q = \frac{-2i\sqrt{R(G^2-p^2)}}{1+\mu\theta_1},$$

$$A_{1,2} = -p \pm i\sqrt{G^2-p^2}, \tag{3.21}$$

and functions θ_1 *and* θ_2 *are regressive. Then every solution of* (3.1) *may be represented in the form* (3.3), (3.9), (3.19), *and error estimate* (2.16) *is satisfied with the function* $M(t)$ *given by* (3.20).

Remark 3.1. For (3.1) with $P_1(t) \equiv 0$, on a continuous time scale ($\mu(t) = 0$), Theorem 3.6 turns to the new version of classical Hartman-Wintner theorem:

Theorem 3.7 (**[16]**)**.** *Assume* $Q(t)$ *is a complex valued function,* $Q \neq 0$, $Q \in C^1$ *and*

$$\int_T^\infty \left| d\left(\frac{Q'(t)}{Q^{3/2}}\right)\right| < \infty, \qquad \gamma = \lim_{t\to\infty}\frac{Q'}{4Q^{3/2}}, \quad \gamma^2 \neq 1, \tag{3.22}$$

and expressions

$$\exp \pm i \int_T^t \sqrt{Q(s) - \left(\frac{Q'(s)}{4Q(s)}\right)^2}\, ds \tag{3.23}$$

are bounded for $t \to \infty$. *Then every solution of* $\psi''(t) + Q\psi(t) = 0$ *could be written in the form* (3.3), (3.4) *with*

$$u_{1,2}(t) = Q^{-1/4} \exp\left(\pm i \int_T^t \sqrt{Q(s) - \left(\frac{Q'(s)}{4Q(s)}\right)^2} ds\right). \tag{3.24}$$

Note that, in our version of this theorem, condition (3.23) is included in condition (3.7), and we have the additional error estimate (2.16) as well.

4. Asymptotic solutions of the self-adjoint second order equations. Consider the selfadjoint second order equation

$$F(\psi) = (a(t)\psi^{\Delta}(t))^{\Delta}(t) + b(t)\psi^{\sigma}(t) = 0, \quad t \in \mathbf{T}_0 \tag{4.1}$$

with complex valued coefficients $a(t), b(t)$. This equation could be deduced from (3.1) by letting

$$P_1^{\sigma}(t) + P_1(t) = \frac{a^{\Delta}(t) + b(t)\mu(t)}{a^{\sigma}(t)}, \tag{4.2}$$

$$Q(t) = \frac{b(t)}{a^{\sigma}(t)}, \qquad L(\psi) = \frac{F(\psi)}{a^{\sigma}}.$$

Introducing auxiliary exponential functions

$$\widetilde{e}_k(t, t_0) = e_{\eta_k/a}(t, t_0), \quad k = 1, 2, \dots, \tag{4.3}$$

define characteristic functions $CF_k(t) = CF(\eta_k)$, $k = 1, 2$, of equation (4.1) as

$$\begin{aligned} CF_k(t) &= \frac{aF(e_{\eta_k/a})}{e_{\eta_k/a}} \\ &= a(t)\eta_k^{\Delta}(t) + \eta_k^{\sigma}(t)\eta_k(t) + a(t)b(t) + \mu(t)b(t)\eta_k(t). \end{aligned} \tag{4.4}$$

Theorem 4.1. *Assume* $a, b \in C_{rd}$, *and there exist complex valued functions* $\eta_1(t), \eta_2(t) \in C^1_{rd}$ *such that conditions*

$$a(t) + \mu(t)\eta_k(t) \neq 0, \quad k = 1, 2, \dots, \tag{4.5}$$

$$\left|\frac{\eta(t)\widetilde{e}_1(t)\widetilde{e}_2(t)}{a(t)}\right| \neq 0, \qquad \eta(t) \equiv \frac{\eta_1(t) - \eta_2(t)}{2}, \quad t > t_0, \tag{4.6}$$

$$M_1(t) \in C_{rd}, \qquad \int_t^\infty M_1(s)\Delta s < \infty, \quad t > t_0, \tag{4.7}$$

are satisfied with

$$M_1(t) = \max_{k,j=1,2} \left| \frac{e_r^{\pm 1}(t) C F_j(t)}{\eta^\sigma(t)(a(t) + \mu(t)\eta_k(t))} \right|, \qquad r(t) = \frac{\eta_2(t) - \eta_1(t)}{a(t) + \mu\eta_1(t)}. \tag{4.8}$$

Then every solution of the equation (4.1) *may be represented in the form* (3.3) *where*

$$u_k(t) = \widetilde{e_k}(t, t_0) = e_{\eta_k/a}(t, t_0), \quad k = 1, 2, \ldots, \tag{4.9}$$

and error estimate (2.16) *is satisfied with the function* $M(t) = M_1(t)$ *given by* (4.8).

Taking

$$\widetilde{e_1} = 1, \qquad \widetilde{e_2}(t) = e_r(t), \tag{4.10}$$
$$r(t) = \frac{1}{a(t)\int_{t_1}^t \frac{e_{\Omega_1}(t,s)\Delta s}{a(s)}}, \qquad \Omega_1(t) = \frac{b(t)\mu(t)}{a(t) - b(t)\mu^2(t)},$$

from Theorem 4.1 we deduce following theorem.

Theorem 4.2. *Assume that real functions* $a, b \in C_{rd}(T, \infty)$ *satisfy the following conditions*

$$a(t) > 0, \qquad a(t) - b(t)\mu^2(t) > 0, \quad t > t_0, \tag{4.11}$$

$$\int_T^\infty M_2(t)\, dt < \infty, \tag{4.12}$$

$$M_2(t) = \max \left| e_r^{\pm 1}(t) b(t) \int_{t_0}^\sigma \frac{e_{\Omega_1}(t, s)\Delta s}{a(s)} \right|. \tag{4.13}$$

Then, for arbitrary constants C_1, C_2, *there exists a solution of* (4.1) *that can be written in the form* (3.3) *with* $u_1(t) = 1$, $u_2(t) = e_r(t)$, *and error estimate* (2.16) *is satisfied with the function* $M(t) = M_2(t)$ *given by* (4.13).

Remark 4.1. If

$$a(t)\int_{t_0}^{t}\frac{e_{\Omega_1}(t,s)\Delta s}{a(s)}\geq t-t_0,\quad t>t_0>0, \tag{4.14}$$

then

$$1\leq e_r(t,t_0)\leq e_{1/(t-t_0)}(t,t_0)=\frac{t-t_0}{t_0},\quad t>t_0>0. \tag{4.15}$$

5. Proofs.

Proof of Theorem 2.1. The substitution $\varphi(t)=\Phi(t)u(t)$ transforms (1.1) into

$$u^{\Delta}(t)=K(t)u(t), \tag{5.1}$$

where $K(t)$ is defined in (2.9). By integration, we get

$$u(t)=C-\int_t^b K(s)u(s)\Delta s,\quad t<s<b, \tag{5.2}$$

or

$$\|u(t)\|\leq\|C\|+\int_t^b\|K(s)\|\cdot\|u(s)\|\Delta s. \tag{5.3}$$

Applying Gronwall's lemma, we have

$$\|u(t)\|\leq\|C\|e_{\|K(t)\|}(b,t), \tag{5.4}$$

where from the definition of an exponential function on a time scale we have

$$e_{\|K\|}(b,t)=\exp\int_t^{\infty}\lim_{m\searrow\mu(s)}\frac{\operatorname{Log}(1+m\|K(s)\|)\Delta s}{m}.$$

From (2.11) and (5.2), we get the following representation

$$\varepsilon(t)=\Phi^{-1}\varphi(t)-C=u(t)-C=-\int_t^b K(s)u(s)\Delta s, \tag{5.5}$$

and the estimate

$$\text{(5.6)}\qquad \begin{aligned} \|\varepsilon(t)\| &\le \int_t^b \|K(s)\|\cdot\|u(s)\|\Delta s \\ &\le \|C\| \int_t^b \|K(s)\| e_{\|K(s)\|}(b,s)\Delta s, \\ \|\varepsilon(t)\| &\le \|C\|[-1+e_{\|K(s)\|}(b,t)], \end{aligned}$$

or (2.12). □

Proof of Theorem 2.2. Invertibility of Φ^σ is followed, from (2.13):

$$\text{(5.7)}\quad \det(\Phi^\sigma) = \begin{vmatrix} e_1^\sigma & e_2^\sigma \\ e_1^\sigma U_1^\sigma & e_2^\sigma U_2^\sigma \end{vmatrix} = e_1^\sigma e_2^\sigma (U_2^\sigma - U_1^\sigma) = -\frac{\theta_{12}^\sigma e_1^\sigma e_2^\sigma}{a_{12}^\sigma} \ne 0.$$

By direct calculations,

$$\text{(5.8)}\qquad \Phi^{-1}(A\Phi - \Phi^\Delta) = \begin{pmatrix} -\frac{CL_1}{2\theta} & \frac{-e_2 CL_2}{2e_1\theta} \\ \frac{e_1 CL_1}{2e_2\theta} & \frac{CL_2}{2\theta} \end{pmatrix},$$

$$\text{(5.9)}\qquad \begin{aligned} K &= (\Phi^{-1})^\sigma (A\Phi - \Phi^\Delta) \\ &= \frac{-a_{12}^\sigma \theta}{a_{12}\theta^\sigma} \begin{pmatrix} \frac{e_1 CL_1}{2\theta e_1^\sigma} & \frac{e_2 CL_2}{2\theta e_1^\sigma} \\ -\frac{e_1 CL_1}{2\theta e_2^\sigma} & -\frac{e_2 CL_2}{2\theta e_2^\sigma} \end{pmatrix}, \end{aligned}$$

$$\text{(5.10)}\qquad \begin{aligned} \|K(t)\| &\le CM(t), \\ M(t) &= \max_{k,j=1,2} \left| \frac{a_{12}^\sigma(t) e_j(t) CL_j(t)}{a_{12}(t)\theta^\sigma(t) e_k^\sigma(t)} \right|. \end{aligned}$$

From condition $M \in C_{rd}$, we have $\|K\| \in C_{rd}$. From Theorem 2.1, it follows that (2.12) is satisfied and, in view of (5.10), (2.16) is satisfied as well if

$$\text{(5.11)}\qquad CL_j = CA_j, \quad j = 1,2.$$

To prove (5.11), we will show that, by the differentiation system, (1.1) could be transformed to the second order equation (3.1).

Indeed, from system (1.1), we have

$$\text{(5.12)}\qquad \varphi_1^\Delta = a_{11}\varphi_1 + a_{12}\varphi_2,$$

$$\text{(5.13)}\qquad \varphi_2^\Delta = a_{21}\varphi_1 + a_{22}\varphi_2.$$

To eliminate φ_2, we find, from (5.12),

$$\varphi_2 = \frac{\varphi_1^\Delta - a_{11}\varphi_1}{a_{12}}, \tag{5.14}$$

and, by substitution into (5.13), we get

$$\left(\frac{\varphi_1^\Delta - a_{11}\varphi_1}{a_{12}}\right)^\Delta = a_{21}\varphi_1 + a_{22}\left(\frac{\varphi_1^\Delta - a_{11}\varphi_1}{a_{12}}\right),$$

or

$$\begin{aligned}\frac{\varphi_1^{\Delta\Delta}a_{12} - \varphi_1^\Delta a_{12}^\Delta}{a_{12}^\sigma a_{12}} - \varphi_1\left(\frac{a_{11}}{a_{12}}\right)^\Delta - \varphi_1^\Delta\left(\frac{a_{11}}{a_{12}}\right)^\sigma \\ = a_{21}\varphi_1 + a_{22}\left(\frac{\varphi_1^\Delta - a_{11}\varphi_1}{a_{12}}\right),\end{aligned}$$

$$\varphi_1^{\Delta\Delta} - \varphi_1^\Delta\left(\frac{a_{12}^\Delta}{a_{12}} + a_{11}^\sigma + \frac{a_{22}a_{12}^\sigma}{a_{12}}\right) + \varphi_1 a_{12}^\sigma\left(\frac{\det(A)}{a_{12}} - \left(\frac{a_{11}}{a_{12}}\right)^\Delta\right) = 0, \tag{5.15}$$

or (3.1) with $\varphi = \psi$, and

$$\begin{aligned}P_1^\sigma(t) + P_1(t) &= -a_{11}^\sigma - \frac{a_{12}^\Delta}{a_{12}} - \frac{a_{22}a_{12}^\sigma}{a_{12}} \\ &= -Tr(A) - \mu a_{11}^\Delta - \frac{a_{12}^\Delta}{a_{12}}(1 + \mu a_{22}),\end{aligned} \tag{5.16}$$

$$Q(t) = \frac{a_{12}^\sigma \det(A)}{a_{12}} - a_{12}^\sigma\left(\frac{a_{11}}{a_{12}}\right)^\Delta. \tag{5.17}$$

From (3.5), we get

(5.18)
$$CL_k(t) = \theta_k^2(t) + (P_1^\sigma(t) + P_1(t))\theta_k(t) + Q(t) + \theta_k^\Delta(t)(1 + \mu(t)\theta_k(t)),$$
$$k = 1, 2, \ldots,$$

$$\begin{aligned}CL_k &= \theta_k^2 - \theta_k\left(\frac{a_{12}^\Delta}{a_{12}} + a_{11}^\sigma + \frac{a_{22}a_{12}^\sigma}{a_{12}}\right)\\ &\quad + \frac{a_{12}^\sigma \det(A)}{a_{12}} - a_{12}^\sigma\left(\frac{a_{11}}{a_{12}}\right)^\Delta + \theta_k^\Delta(1+\mu\theta_k)\\ &= \theta_k^2 - \theta_k\left(\frac{a_{12}^\Delta}{a_{12}} + Tr(A) + \mu a_{11}^\Delta + \frac{\mu a_{22}a_{12}^\Delta}{a_{12}}\right)\\ &\quad + \frac{a_{12}^\sigma \det(A)}{a_{12}} - a_{12}^\sigma\left(\frac{a_{11}}{a_{12}}\right)^\Delta + \theta_k^\Delta(1+\mu\theta_k)\end{aligned}$$

or, in view of (2.1), and

$$\theta_k = a_{12}U_k + a_{11}, \qquad \theta_k^\Delta = a_{12}^\Delta U_k + a_{12}^\sigma U_k^\Delta + a_{11}^\Delta$$

we get (5.11):

$$CL_k = \frac{a_{12}^\sigma[\theta_k^2 - \theta_k Tr(A) + \det(A)]}{a_{12}} + a_{12}^\sigma(1+\mu\theta_k)U_k^\Delta = CA_k. \qquad \square \tag{5.19}$$

From (2.2), (3.5) and (5.18)

$$\begin{aligned}HA(t) = HL(t) &= \frac{CL_1 - CL_2}{\theta_1 - \theta_2}\\ &= \mu\theta_1^\Delta + \frac{\theta^\sigma + \theta}{\theta}\theta_1 + \frac{\theta^\Delta}{\theta} - 2\theta^\sigma\\ &\quad + P_1^\sigma + P_1,\end{aligned} \tag{5.20}$$

we get (2.3):

$$\begin{aligned}HA(t) = HL(t) &= \frac{\theta^\sigma + \theta}{\theta}\theta_1 + \frac{\theta^\Delta}{\theta} - 2\theta^\sigma - Tr(A)\\ &\quad - \frac{a_{12}^\Delta}{a_{12}}(1+\mu a_{22}) + \mu(\theta_1^\Delta - a_{11}^\Delta),\end{aligned}$$

or

$$HL(t) = \mu\theta_1^\Delta + \frac{\theta^\sigma + \theta}{\theta}\theta_1 + \frac{\theta^\Delta}{\theta} - 2\theta^\sigma - a_{11}^\sigma - \frac{a_{12}^\Delta}{a_{12}} - \frac{a_{12}^\sigma a_{22}}{a_{12}}. \tag{5.21}$$

Proof of Remark 2.1. To prove (2.36), consider equation (2.37):

$$P_1(t) + P_1^\sigma(t) = 2P(t).$$

If $\mu(t) = 0$, this formula means $P_1 = P$. If $\mu(t) \neq 0$, from $P_1^\sigma = P_1 + \mu P_1^\Delta$, we get

$$P_1^\Delta(t) = \frac{2P(t)}{\mu} - \frac{2P_1(t)}{\mu(t)}.$$

Solving this first order linear equation for P_1, we get (2.36).

Indeed, it is known ([**7**]) that the solutions of the linear equation

$$y^\Delta(t) = p(t)y + f(t)$$

are given by the formula

$$\begin{aligned} y(t) &= y_0(t)e_p(t,t_0) + \int_{t_0}^t e_p(t,\sigma(s))f(s)\Delta s \\ &= y_0(t)e_p(t,t_0) + \int_{t_0}^t \frac{e_p(t,s)f(s)\Delta s}{1+\mu(s)p(s)}. \end{aligned} \qquad \square$$

Lemma 5.1. *If $\theta \in C_{rd}^1$ and*

(5.22) $\theta_1(t)$

$$= \begin{cases} \theta(t)\int_{t_0}^t e_{-2/\mu}(t,\sigma(\tau))\left[\frac{2}{\mu} + \frac{1}{\mu}\left(\frac{1}{\theta}\right)^\Delta - \frac{P_1^\sigma + P_1}{\mu\theta^\sigma}\right](\tau)\Delta\tau & \mu(t) \neq 0, \\ \theta(t) - \frac{\theta'(t)}{2\theta(t)} - P_1(t) & \mu(t) \equiv 0, \end{cases}$$

or

$$\frac{1}{2\theta(t)} = -\int_{t_0}^t \frac{e_\Lambda(t,\sigma(s))\Delta s}{1-\mu(s)[\theta_1^\sigma(s) + P_1(s) + P_1^\sigma(s)]}, \tag{5.23}$$

$$\Lambda(t) = \frac{\theta_1(t) + \theta_1^\sigma(t) + P_1^\sigma(t) + P_1(t)}{1-\mu(t)[\theta_1^\sigma(t) + P_1(t) + P_1^\sigma(t)]}, \quad 1 + \mu\Lambda(t) \neq 0, \tag{5.24}$$

then

$$HA(t) = HL(t) = 0, \quad \textit{for all } t \in \mathbf{T}_0.$$

Remark 5.1. By substitution,

$$\theta_j = \xi_j - P_1, \qquad \xi = \frac{\xi_1 - \xi_2}{2} \quad j = 1,2 \tag{5.25}$$

one can simplify formulas (3.5) for characteristic function $CL_k(t)$ and function HL:

$$CL_k(t) = \xi_k^\Delta(1-\mu P_1) + \xi_k^\sigma \xi_k + R(t), \tag{5.26}$$
$$R(t) = Q(t) - P_1^2(t) - P_1^\Delta(t), \quad k = 1, 2, \dots,$$

$$HL(t) = \frac{\xi^\Delta}{\xi} - 2\xi^\sigma + P_1\left(1 - \frac{\xi^\sigma}{\xi}\right) + \xi_1^\sigma + \frac{\xi^\sigma}{\xi}\xi_1, \tag{5.27}$$

Indeed, formulas (5.23) and (5.24) turn into

$$\xi_1(t) = \begin{cases} P_1 + \xi \int_{t_0}^t e_{-2/\mu}(t,\sigma(\tau))\left(\frac{2}{\mu} + \frac{1}{\mu}\left(\frac{1}{\xi}\right)^\Delta - \frac{P_1^\sigma + P_1}{\mu\xi^\sigma}\right)\Delta\tau & \mu \neq 0, \\ \xi(t) - \frac{\xi'(t)}{2\xi(t)} & \mu \equiv 0, \end{cases} \tag{5.28}$$

$$\frac{1}{2\xi(t)} = -\int_{t_0}^t \frac{e_\Lambda(t,\sigma(s))\Delta s}{1-\mu(\xi_1^\sigma + P_1)}, \quad 1 + \mu(t)\Lambda(t) \neq 0, \tag{5.29}$$
$$\Lambda(t) = \frac{\xi_1 + \xi_1^\sigma}{1 - \mu(\xi_1^\sigma + P_1)}.$$

Further, from

$$\begin{aligned} CL_k &= \theta_k^\sigma\theta_k + (P_1^\sigma + P_1)\theta_k + Q + \theta_k^\Delta \\ &= (\xi_k - P_1)^\sigma(\xi_k - P_1) + (P_1 + P_1^\sigma)(\xi_k - P_1) \\ &\quad + R + P_1^2 + P_1^\Delta + (\xi_k - P_1)^\Delta, \end{aligned}$$

we get

$$CL_k = \xi_k^2 + R + \xi_k^\Delta(1 + \mu\xi_k - \mu P_1). \tag{5.30}$$

or (5.26).

Remark 5.2. By another substitution,

$$\theta_j = \xi_j - P, \quad j = 1, 2 \tag{5.31}$$

characteristic function CL_k turns into

$$CL_j = \xi_j^\Delta(1-\mu P) + \xi_j(\xi_j^\sigma - \mu P^\Delta) + R_1, \tag{5.32}$$
$$R_1 = Q - P^2 - P^\Delta + \mu P^\Delta P, \quad j = 1, 2, \dots .$$

Formulas (5.23) and (5.24) in turn give us

$$\xi_1(t) = \begin{cases} P(t) + \xi \int_{t_0}^t e_{-2/\mu}(t,\sigma(s)) \left[\frac{2}{\mu} + \frac{1}{\mu}\left(\frac{1}{\xi}\right)^{\Delta} - \frac{2P}{\mu\xi^\sigma}\right](s)\Delta s & \mu \neq 0, \\ \xi(t) - \frac{\xi'(t)}{2\xi(t)} & \mu \equiv 0, \end{cases} \tag{5.33}$$

$$\frac{1}{\xi(t)} = -\int_{t_0}^t \frac{e_\Lambda(t,\sigma(s))\Delta s}{1-\mu\xi_1^\sigma - (2P-P^\sigma)\mu}, \quad 1+\mu\Lambda \neq 0, \tag{5.34}$$
$$\Lambda(t) = \frac{\xi_1 + \xi_1^\sigma + P - P^\sigma}{1-\mu\xi_1^\sigma - (2P-P^\sigma)\mu}.$$

Proof of Lemma 5.1. From (2.37) and (5.20), assuming $\mu \neq 0$, we have

$$\frac{HL}{\mu\theta^\sigma} = \frac{\theta_1^\Delta}{\theta^\sigma} + \frac{(\theta^\sigma+\theta)\theta_1}{\mu\theta^\sigma\theta} + \frac{\theta^\Delta}{\mu\theta^\sigma\theta} + \frac{2P}{\mu\theta^\sigma} - \frac{2}{\mu} \tag{5.35}$$

$$\frac{HL}{\mu\theta^\sigma} = \frac{\theta_1^\Delta}{\theta^\sigma} + \frac{\theta_1}{\mu\theta^\sigma} + \frac{\theta_1}{\mu\theta} - Y_1 = \frac{1}{\mu}\left(\frac{\theta_1}{\theta}\right)^\sigma + \frac{\theta_1}{\mu\theta} - Y_1 \tag{5.36}$$

where

$$Y_1 = \frac{2}{\mu} - \frac{2P}{\mu\theta^\sigma} - \frac{\theta^\Delta}{\mu\theta^\sigma\theta} = \frac{2}{\mu} + \frac{1}{\mu}\left(\frac{1}{\theta}\right)^\Delta - \frac{2P}{\mu\theta^\sigma}. \tag{5.37}$$

Solving $HA = 0$, $(HL)/\mu\theta^\sigma = 0$ or

$$\left(\frac{\theta_1(t)}{\theta(t)}\right)^\Delta + \frac{2\theta_1(t)}{\mu(t)\theta(t)} = Y_1(t), \tag{5.38}$$

for θ_1, we get (5.23) in the case $\mu \neq 0$. The case $\mu \equiv 0$ is obvious.

Further, again from (5.20), we have

$$\frac{HL}{\theta^\sigma} = \frac{\mu\theta_1^\Delta}{\theta^\sigma} + \frac{\theta_1}{\theta} + \frac{\theta_1}{\theta^\sigma} + \frac{\theta^\Delta}{\theta^\sigma\theta} - 2 + \frac{2P}{\theta^\sigma} \tag{5.39}$$
$$= \frac{\theta_1}{\theta} - 2 + \frac{\theta_1^\sigma + 2P}{\theta^\sigma} - \left(\frac{1}{\theta}\right)^\Delta$$

and, using the formula

$$\frac{1}{\theta^\sigma} = \frac{1}{\theta} + \mu\left(\frac{1}{\theta}\right)^\Delta, \tag{5.40}$$

we get equation

$$\frac{HL}{\theta^\sigma} = \left(\frac{1}{\theta}\right)^\Delta (\mu\theta_1^\sigma + 2P\mu - 1) + \frac{1}{\theta}(\theta_1 + \theta_1^\sigma + 2P) - 2 = 0, \tag{5.41}$$

or

$$\left(\frac{1}{\theta(t)}\right)^\Delta - \frac{\Lambda(t)}{\theta(t)} = \frac{2}{\mu(t)\theta_1^\sigma(t) + 2P(t)\mu(t) - 1}, \tag{5.42}$$

with solution

$$\frac{1}{\theta(t)} = -\int_{t_0}^{t} \frac{2e_\Lambda(t,\sigma(s))\Delta s}{1-\mu\theta_1^\sigma - 2P\mu}, \qquad \Lambda(t) = \frac{\theta_1 + \theta_1^\sigma + 2P}{1-\mu\theta_1^\sigma - 2P\mu}. \tag{5.43}$$ $\square$

From (3.5), we may consider the characteristic function formally as an eigenvalue of an operator L:

$$Le_\theta(t,t_0) = CL(\theta)e_\theta(t,t_0). \tag{5.44}$$

Proof of Theorem 2.3. By choosing $\xi_1 = 0$, we deduce Theorem 2.3 from Theorem 2.2. From (5.29), $\Lambda = 0$, and since $1 - P_1\mu \neq 0$, from (5.29), we get

$$\begin{aligned} \xi_2 &= \xi_2 - \xi_1 = \frac{1}{\int_{t_0}^{t} \frac{\Delta s}{1-P_1(s)\mu(s)}}, \\ \theta_1 &= -P_1, \\ \theta_2 &= \frac{1}{\int_{t_0}^{t} \frac{\Delta s}{1-P_1(s)\mu(s)}} - P_1(t). \end{aligned} \tag{5.45}$$

Using the Kronecker symbol

$$\delta_{j2} = \begin{cases} 0 & j = 1, \\ 1 & j = 2, \end{cases}$$

we can rewrite

$$\xi_j = \frac{\delta_{j2}}{\int_{t_0}^{t} \Delta s/(1-P_1\mu)}, \quad j = 1, 2.$$

$$1+\mu\theta_1 = 1+\mu\xi_1 - P_1\mu = 1 - P_1\mu,$$
$$1+\mu\theta_2 = 1 - P_1\mu + \frac{\mu}{\int_{t_0}^t \Delta s/(1-P_1\mu)},$$

$$1+\mu\theta_j = 1 - P_1\mu + \frac{\mu\delta_{j2}}{\int_{t_0}^t \Delta s/(1-P_1\mu)}, \quad \frac{1}{\xi^\sigma} = \int_{t_0}^\sigma \frac{\Delta s}{1-P_1\mu}. \tag{5.46}$$

From Lemma 5.1 and (5.26), we get

$$HA \equiv 0, \quad CL_1 = CL_2 = CA_1 = CA_2 = R. \tag{5.47}$$

Condition (2.13) here means

$$\left|\frac{\theta e_1 e_2}{a_{12}}\right| = \left|\frac{\xi e_1 e_2}{a_{12}}\right| = \left|\frac{e_1(t)e_2(t)}{a_{12}(t)\int_{t_0}^t \Delta s/(1-P_1\mu(s))}\right| \neq 0. \tag{5.48}$$

Using the circle product and Euler exponent rules [**7**]

$$p \oplus q = p + q + \mu pq, \qquad p \ominus q = \frac{p-q}{1+\mu q}, \tag{5.49}$$
$$e_2 e_1 = e_{\theta_1 \oplus \theta_2}, \qquad \frac{e_2}{e_1} = e_{\theta_2 \ominus \theta_1},$$

we get

$$e_q(t) = \frac{e_2(t)}{e_1(t)}, \tag{5.50}$$
$$q(t) = \frac{\theta_2(t) - \theta_1(t)}{1+\mu(t)\theta_1(t)} = \frac{1}{(1-P_1\mu(t))\int_{t_0}^t \Delta s/(1-P_1(s)\mu(s))}.$$

From (2.15), we obtain (2.20):

$$M(t) = \max_{j=1,2}\left|\frac{e_q^{\pm 1}(t)R(t)a_{12}^\sigma(t)}{a_{12}(t)(1+\mu\theta_j(t))}\int_{t_0}^\sigma \frac{\Delta s}{1-P_1(s)\mu(s)}\right|. \qquad \square$$

Proof of Theorem 2.4. Choosing linear equation approximation

$$CL_1 \approx \xi_1^\Delta(1-\mu P_1) + R = 0,$$

we have by using formula (5.29):

$$
\begin{aligned}
\xi_1 &= -\int_{t_0}^{t} \frac{R(s)\Delta s}{1-\mu P_1},\\
\xi_2 &= \xi_1 - (\xi_1 - \xi_2)\\
&= -\int_{t_0}^{t} \frac{R(s)\Delta s}{1-\mu P_1} + \frac{1}{\int_{t_0}^{t}(e_\Lambda(t,\sigma(s))\Delta s)/(1-\mu(P_1+\xi_1^\sigma))}\\
\theta_1 &= -\int_{t_0}^{t} \frac{R(s)\Delta s}{1-\mu P_1} - P_1,\\
\theta_2 &= \frac{1}{\int_{t_0}^{t}(e_\Lambda(t,\sigma(s))\Delta s)/(1-\mu(P_1+\xi_1^\sigma))} - \int_{t_0}^{t} \frac{R(s)\Delta s}{1-\mu P_1} - P_1,
\end{aligned}
$$

where Λ is as in (5.29), and we have $HL \equiv 0$ from Lemma 5.1.

Further,

$$CL_1 = CL_2 = \xi_1^\Delta(1-\mu P_1) + \xi_1^\sigma \xi_1 + R = \xi_1^\sigma \xi_1,$$

$$e_q(t) = \frac{e_2(t)}{e_1(t)}, \qquad q = \frac{\theta_2-\theta_1}{1+\theta_1\mu} = \frac{1}{(1+\theta_1\mu)\int_{t_0}^{t}\frac{e_\Lambda(t,\sigma(s))\Delta s}{1-\mu(P_1+\xi_1^\sigma)}},$$

$$
\begin{aligned}
M(t) &= \max_{k,j=1,2}\left|\frac{a_{12}^\sigma(t)e_q^{\pm 1}(t)CL_k(t)}{a_{12}(t)2\theta^\sigma(t)(1+\mu(t)\theta_j(t))}\right|\\
&= \max\left|\frac{a_{12}^\sigma(t)e_q^{\pm 1}(t)}{a_{12}(1+\mu\theta_j(t))}\int_{t_0}^{t}\frac{R(s)\Delta s}{1-\mu(s)P_1(s)}\right.\\
&\qquad \left.\times\int_{\sigma_0}^{\sigma}\frac{R(s)\Delta s}{1-\mu(s)P_1(s)}\int_{t_0}^{t}\frac{e_\Lambda(t,\sigma(s))\Delta s}{1-\mu(P_1+\xi_1^\sigma)(s)}\right|. \qquad \square
\end{aligned}
$$

Proof of Theorem 3.1. Theorem 3.1 is a direct consequence of Theorem 2.2 applied to the system:

$$\begin{pmatrix}\psi(t)\\ \psi^\Delta(t)\end{pmatrix}^{\Delta} = \begin{pmatrix}0 & 1\\ -Q(t) & -P_1(t)-P_1^\sigma(t)\end{pmatrix}\begin{pmatrix}\psi(t)\\ \psi^\Delta(t)\end{pmatrix} \tag{5.51}$$

with $a_{12} \equiv 1$, $a_{11} \equiv 0$, $a_{21} \equiv -Q(t)$, $a_{22} \equiv -P_1(t) - P_1^\sigma(t)$, and fundamental matrix (2.7) with $U_j(t) = \theta_j(t)$. Note that representation (2.11) becomes (3.3). $\square$

Proof of Theorem 3.2. Theorem 3.2 follows from Theorem 3.1 by choosing phase functions (3.10):

$$\theta_1(t) = -P_1(t), \quad \theta_2(t) = \frac{1}{\int_{t_0}^t (2\Delta s)/(1-P_1(s)\mu(s))} - P_1(t), \tag{5.52}$$

$$e_q(t) = \frac{e_2(t)}{e_1(t)}, \tag{5.53}$$
$$q = \frac{\theta_2 - \theta_1}{1+\mu\theta_1} = \frac{1}{(1-P_1(t)\mu(t)) \int_{t_0}^t (2\Delta s)/(1-P_1(s)\mu(s))}.$$

From (3.8), (5.47) and (5.52), we get (3.11):

$$M(t) = \max_{k,j=1,2} \left| \frac{e_k(t) CL_k(t)}{e_j(t)\theta^\sigma(t)(1+\mu(t)\theta_j(t))} \right|$$
$$= \max_{j=1,2} \left| \frac{e_q^{\pm 1}(t) R(t)}{1+\mu\theta_j(t)} \int_{t_0}^\sigma \frac{\Delta s}{1-P_1(s)\mu(s)} \right|.$$

□

Proof of Theorem 3.3. We deduce Theorem 3.3 from Theorem 3.1 by choosing

$$\xi_1 = im, \qquad \xi_2 = -im, \qquad m = \text{constant} > 0. \tag{5.54}$$

Then

$$\xi = im, \qquad \theta_1 = im - P_1(t), \qquad \theta_2 = -im - P_1(t), \tag{5.55}$$

and, from (5.27),

$$HL = \frac{\xi^\Delta}{\xi} - 2\xi^\sigma + P_1\left(1 - \frac{\xi^\sigma}{\xi}\right) + \xi_1^\sigma + \frac{\xi^\sigma}{\xi}\xi_1 \equiv 0. \tag{5.56}$$

From (5.30),

$$CL_1(t) = CL_2(t) = \xi_1^2 + R + \xi_1^\Delta(1+\mu\xi_1 - \mu P_1) = R(t) - m^2. \tag{5.57}$$

From (3.8),

$$M(t) = \max_{k,j=1,2} \left| \frac{e_k(t) CL_k(t)}{e_j(t)\theta^\sigma(t)(1+\mu\theta_j)} \right| = \max_{j=1,2} \left| \frac{e_q^{\pm 1}(t)(R(t)-m^2)}{m(1+\mu\theta_j(t))} \right|, \tag{5.58}$$

$$q = \frac{\theta_2 - \theta_1}{1 + \theta_1 \mu} = \frac{-2im}{1 + im\mu - P_1\mu}. \tag{5.59}$$ □

Proof of Theorem 3.4. We deduce Theorem 3.4 from Theorem 3.1 by choosing

$$\begin{aligned} \xi_1 &= i\sqrt{R}, \qquad \xi_2 = -i\sqrt{R}, \\ \theta_1 &= i\sqrt{R} - P_1, \qquad \theta_2 = -i\sqrt{R} - P_1. \end{aligned} \tag{5.60}$$

We have, from (5.30),

$$\begin{aligned} CL_1 &= \xi_1^\Delta(1 + \mu\xi_1 - \mu P_1) + \xi_1^2 + R = i\sqrt{R}^\Delta(1 + i\sqrt{R}\mu - \mu P_1) \\ &= \frac{i(\sqrt{R^\sigma} - \sqrt{R})(1 + i\sqrt{R}\mu - \mu P_1)}{\mu}, \end{aligned}$$

$$\frac{CL_1}{2\theta^\sigma} = \frac{1}{2\mu}\left(1 - \sqrt{\frac{R}{R^\sigma}}\right)(1 + i\mu\sqrt{R} - \mu P_1), \tag{5.61}$$

$$CL_2 = -i\sqrt{R}^\Delta(1 - i\sqrt{R}\mu - \mu P_1) = -\frac{i(\sqrt{R^\sigma} - \sqrt{R})(1 - i\sqrt{R}\mu - \mu P_1)}{\mu},$$

$$\frac{CL_2}{2\theta^\sigma} = -\frac{1}{2\mu}\left(1 - \sqrt{\frac{R}{R^\sigma}}\right)(1 - i\mu\sqrt{R} - \mu P_1). \tag{5.62}$$

From (3.8),

$$M_{1,2} = \left| \frac{e_q^{\pm 1}}{2\mu(1 + \mu\theta_j)} \left(1 - \sqrt{\frac{R}{R^\sigma}}\right)(1 \pm i\mu\sqrt{R} - \mu P_1) \right|, \tag{5.63}$$

$$q = \frac{\theta_2 - \theta_1}{1 + \theta_1\mu} = \frac{-2i\sqrt{R}}{1 + i\sqrt{R}\mu - P_1\mu}. \tag{5.64}$$ □

Proof of Theorem 3.5. We deduce Theorem 3.5 from Theorem 3.1. Choosing

$$\begin{aligned} \xi_1(t) &= \int_{t_0}^t \frac{R(s)\Delta s}{P_1(s)\mu(s) - 1}, \\ \xi_2 &= \xi_1 - \frac{1}{\int_{t_0}^t (e_\Lambda(t, \sigma(s))\Delta s)/(\mu\xi_1^\sigma(s) + P_1(s)\mu(s) - 1)}, \end{aligned} \tag{5.65}$$

we get from (5.25) and (5.29)

$$\theta_j(t) = \xi_j(t) - P_1(t), \quad j = 1, 2, \tag{5.66}$$
$$\Lambda(t) = \frac{\xi_1 + \xi_1^\sigma}{1 - \mu\xi_1^\sigma - P_1\mu} = \frac{\theta_1 + \theta_1^\sigma + 2P}{1 - \mu\theta_1^\sigma - 2P\mu}.$$

Using (5.29),

$$\frac{1}{2\theta(t)} = \frac{1}{2\xi(t)} = \frac{1}{\xi_1(t) - \xi_2(t)} = \int_{t_0}^{t} \frac{e_\Lambda(t, \sigma(s))\Delta s}{\mu\xi_1^\sigma(s) + P_1(s)\mu(s) - 1}, \tag{5.67}$$

from Lemma 5.1 we get $HL = 0$ and $CL_1 = CL_2$.

From (5.26), we get

$$CL_1 = CL_2 = \xi_1^\Delta(1 - P_1\mu) + \xi_1^\sigma\xi_1 + R = \xi_1^\sigma\xi_1. \tag{5.68}$$

From (3.8), we get

$$M(t) = \max\left|\frac{e_q^{\pm 1}(t)CL_k(t)}{2\theta^\sigma(1 + \mu\theta_j)}\right| \tag{5.69}$$
$$= \max\left|\frac{\xi_1^\sigma(t)\xi_1(t)e_q^{\pm 1}(t)}{1 + \mu\theta_j(t)} \int_{t_0}^{t} \frac{e_\Lambda(t, \sigma(s))\Delta s}{1 - \mu(s)[\xi_1^\sigma(s) + P_1(s)]}\right|, \tag{5.70}$$
$$q = \frac{\theta_2 - \theta_1}{1 + \theta_1\mu} = \frac{\left(\int_{t_0}^{t} \frac{e_\Lambda(t,\sigma(s))\Delta s}{1-\mu(s)[\xi_1^\sigma(s)+P_1(s)]}\right)^{-1}}{1 + \mu(t)\left(\int_{t_0}^{t} \frac{R\Delta s}{P_1(s)\mu(s)-1} - P_1(t)\right)}. \qquad \square$$

Proof of Theorem 3.6. We deduce Theorem 3.6 from Theorem 3.1 by choosing Hartman-Wintner approximations of (3.1), that is,

$$\xi_1(t) = A(t)\sqrt{R(t)}, \qquad G(t) = \left(\frac{R(t)}{R^\sigma(t)}\right)^{1/4}. \tag{5.71}$$

In the case $\mu \neq 0$, we have

$$(\sqrt{R})^\Delta = \frac{\sqrt{R^\sigma} - \sqrt{R}}{\mu} = \frac{R^\sigma - R}{\mu\sqrt{R}(1 + \sqrt{R^\sigma/R})} = \frac{R^\Delta}{\sqrt{R}(1 + G^{-2})}, \tag{5.72}$$

and

$$\xi_1^\Delta = A\sqrt{R}^\Delta + A^\Delta\sqrt{R^\sigma} = \frac{AG^2R^\Delta}{\sqrt{R}(1 + G^2)} + \frac{A^\Delta\sqrt{R}}{G^2}. \tag{5.73}$$

Further,

$$
\begin{aligned}
CL_1 &= \xi_1^\sigma \xi_1 + R + \xi_1^\Delta (1 - P_1\mu) \\
&= A^\sigma A \sqrt{R^\sigma R} + R + (1 - P_1\mu)\left(\frac{AG^2R^\Delta}{\sqrt{R}(1+G^2)} + \frac{A^\Delta\sqrt{R}}{G^2}\right) \\
&= (A^2 + \mu A^\Delta A)\frac{R}{G^2} + R + (1 - P_1\mu)\left(\frac{AG^2R^\Delta}{\sqrt{R}(1+G^2)} + \frac{A^\Delta\sqrt{R}}{G^2}\right),
\end{aligned} \tag{5.74}
$$

or

$$
CL_1 = \frac{R}{G^2}\left(A^2 + 2pA + G^2 + \frac{A^\Delta}{\sqrt{R}}(1 - P_1\mu + \mu A\sqrt{R})\right) \tag{5.75}
$$

where

$$
p = \frac{(1 - P_1\mu)G^4R^\Delta}{2R^{3/2}(1+G^2)}. \tag{5.76}
$$

Choosing A for the solutions of a quadratic equation,

$$
A^2 + 2pA + G^2 = 0, \qquad A_{1,2} = -p \pm i\sqrt{G^2 - p^2}, \tag{5.77}
$$

we get

$$
CL_j = \frac{A_j^\Delta R}{G^2\sqrt{R}}(1 - P_1\mu + \mu A_j\sqrt{R}). \tag{5.78}
$$

Thus, we have

$$
\begin{aligned}
\xi_1 = A_1\sqrt{R}, \qquad \xi_2 = A_2\sqrt{R}, \\
\xi_1 - \xi_2 = (A_1 - A_2)\sqrt{R} = 2i\sqrt{R(G^2 - p^2)},
\end{aligned} \tag{5.79}
$$

$$
\begin{aligned}
\theta_1 &= i\sqrt{R(G^2 - p^2)} - p\sqrt{R} - P_1, \\
\theta_2 &= -i\sqrt{R(G^2 - p^2)} - p\sqrt{R} - P_1,
\end{aligned} \tag{5.80}
$$

$$
\frac{CL_j}{\xi_1 - \xi_2} = \frac{CL_j}{2i\sqrt{R(G^2 - p^2)}} = \frac{A_j^\Delta(1 - P_1\mu + \mu A_j\sqrt{R})}{2iG^2\sqrt{G^2 - p^2)}}, \tag{5.81}
$$

$$
q = \frac{\theta_2 - \theta_1}{1 + \mu\theta_1} = \frac{-2i\sqrt{R(G^2 - p^2)}}{1 - \mu P_1 + i\mu\sqrt{R(G^2 - p^2)} - \mu p\sqrt{R}}, \tag{5.82}
$$

$$M(t) = \max_{k,j=1,2} \left| \frac{e_q^{\pm 1} CL_k}{2\theta^\sigma(1+\mu\theta_j)} \right| = \max_{j=1,2} \left| \frac{e_q^{\pm 1}\sqrt{R} A_j^\Delta (1 - P_1\mu + \mu A_j\sqrt{R})}{2G^2(1+\mu\theta_j)\sqrt{R^\sigma (G^\sigma)^2 - (p^\sigma)^2}} \right|. \tag{5.83}$$

or (3.20):

$$M(t) = \max_{j=1,2} \left| \frac{e_q^{\pm 1} A_j^\Delta (1 - P_1\mu + \mu A_j\sqrt{R})}{2(1+\mu\theta_j)\sqrt{(G^2 - p^2)^\sigma}} \right|.$$

In the case $\mu = 0$, by similar calculations, we have

$$G \equiv 1, \qquad p = \frac{R'(t)}{4R^{3/2}}, \qquad q = -2i\sqrt{R(1-p^2)},$$
$$CL_1 = R(A^2 + 2pA + 1 + AR^{-1/2}),$$
$$A_{1,2} = -p \pm i\sqrt{1-p^2}$$
$$\xi_{1,2} = A_{1,2}\sqrt{R} = -\frac{R'}{4R} \pm i\sqrt{R(1-p^2)},$$

and formulas (5.81)–(5.83) are true for the case $\mu = 0$ as well.

In the case (compare with Theorem 3.7)

$$\mu(t) \equiv 0, \qquad P_1(t) \equiv 0,$$

we have

$$R(t) = Q(t), \qquad G(t) = 1,$$
$$p = \frac{Q'(t)}{Q^{3/2}}, \qquad q = -2i\sqrt{Q(1-p^2)},$$

$$\theta_{1,2}(t) = \pm i\sqrt{Q(1-p^2)} - \frac{Q'}{4Q}, \qquad A_1 = -p \pm i\sqrt{1-p^2},$$
$$M(t) = \left| \frac{A'(t)}{2\sqrt{1-p^2}} \right| \le C|p'(t)|.$$

Condition (3.23) means that

$$|e_q^{\pm 1}(t)| \le \text{const},$$

and, under this condition, the expression for $M(t)$ will be simpler. $\square$

Proof of Example 2.1. For equation (2.17), we have

$$P(t) = P_1(t) \equiv 0, \qquad R(t) = Q(t).$$

Choosing phase functions as in (3.10)

$$\theta_1 = 0, \qquad \theta_2 = \frac{1}{t - t_0} = q(t),$$
$$u_1 = e_{\theta_1} = 1, \qquad u_2 = e_{\theta_2} = e_{1/(t-t_0)}(t, t_0) = \frac{t - t_0}{t_0},$$

and, taking $t_0 > 0$, we get

$$\left| \frac{e_{\theta_1}(t) e_{\theta_2}(t)}{2(t - t_0)} \right| = \frac{t - t_0}{2t_0(t - t_0)} = \frac{1}{2t_0} > 0,$$

and condition (3.6) is satisfied.

From (3.11), we get

$$M(t) = \max \left| \frac{(\sigma(t) - t_0) Q(t) e_q^{\pm 1}(t, t_0)}{1 + \mu(t)\theta_j(t)} \right| \le Ct\sigma(t)|Q(t)|. \qquad \square$$

Lemma 5.2. *If*

$$P_1(t) + P_1^\sigma(t) = \frac{a^\Delta(t) + b(t)\mu(t)}{a^\sigma(t)}, \qquad Q(t) = \frac{b(t)}{a^\sigma(t)}, \tag{5.84}$$
$$\theta_k(t) = \frac{\eta_k(t)}{a(t)}, \quad k = 1, 2,$$

then

$$L[\psi] = \frac{F[\psi]}{a^\sigma(t)}, \qquad CL_k(t) = \frac{CF_k(t)}{a^\sigma(t) a(t)}. \tag{5.85}$$

Proof of Lemma 5.2. Indeed,

$$\frac{F[\psi]}{a^\sigma} = \frac{a^\Delta \psi^\Delta + a^\sigma \psi^{\Delta\Delta} + b\psi^\sigma}{a^\sigma}$$
$$= \psi^{\Delta\Delta} + \frac{a^\Delta + b\mu}{a^\sigma}\psi^\Delta + \frac{b}{a^\sigma}\psi = L[\psi],$$

since

$$L[\psi] = \psi^{\Delta\Delta}(t) + (P_1 + P_1^\sigma)\psi^\Delta(t) + Q(t)\psi(t) = 0, \quad t > t_0, \tag{5.86}$$

$$CL_1(t) = \frac{L[e_{\theta_1}]}{e_{\theta_1}} = \frac{F[e_{\theta_1}]}{a^\sigma e_{\theta_1}} = \frac{F[e_{\eta_1/a}]}{a^\sigma e_{\eta_1/a}} = \frac{CF_1(t)}{a^\sigma a(t)}. \qquad \square$$

Lemma 5.3. *If $\eta \in C^1_{rd}$ and*

(5.87)
$$\eta_1(t) = \begin{cases} \eta(t)\int_{t_2}^t \left[\frac{a-b\mu^2}{\mu}\left(\frac{1}{\eta}\right)^\Delta + \frac{2}{\mu} - \frac{b}{\eta}\right] e_{-2/\mu}(t,\sigma(s))\Delta s & \mu(t) \neq 0, \\ \eta(t) - \frac{a\eta'(t)}{2\eta(t)} & \mu(t) \equiv 0, \end{cases}$$

or

(5.88)
$$\frac{1}{\eta_1(t)-\eta_2(t)} = -\int_{t_1}^t \frac{e_\Omega(t,\sigma(s))\Delta s}{a-\eta_1^\sigma\mu - b\mu^2},$$
$$\Omega(t) = \frac{\eta_1+\eta_1^\sigma + b\mu}{a-\eta_1^\sigma\mu - b\mu^2}, \quad 1+\mu(t)\Omega(t) \neq 0.$$

Then

(5.89)
$$HF(t) = \frac{CF_1(t)-CF_2(t)}{a^\sigma(t)(\eta_1(t)-\eta_2(t))} = HL(t) \equiv 0.$$

Proof of Lemma 5.3. Denoting $\eta_{12} = \eta_1 - \eta_2$, in view of $\eta_2 = \eta_1 - 2\eta$, we get

$$HL = \frac{CL_1 - CL_2}{\theta_1-\theta_2} = \frac{CF_1-CF_2}{a^\sigma a(\theta_1-\theta_2)} = \frac{CF_1-CF_2}{a^\sigma(\eta_1-\eta_2)}$$
$$= HF = \frac{a\eta_{12}^\Delta + \eta_1^\sigma\eta_1 - \eta_2^\sigma\eta_2 + b\mu\eta_{12}}{a^\sigma\eta_{12}}$$
$$HF = \frac{1}{a^\sigma}\left(\frac{a\eta^\Delta}{\eta} + \eta_1^\sigma + \frac{\eta^\sigma}{\eta}\eta_1 - 2\eta^\sigma + b\mu\right)$$

In the case $\mu \equiv 0$, from (5.87), we get $HF \equiv 0$.

Assume $\mu > 0$. To solve equation $HF = 0$ for η_1, from

$$\frac{a^\sigma HF}{\eta^\sigma} = -a\left(\frac{1}{\eta}\right)^\Delta + \frac{\eta_1^\sigma}{\eta^\sigma} + \frac{\eta_1}{\eta} - 2 + \frac{b\mu}{\eta^\sigma}$$
$$= -a\left(\frac{1}{\eta}\right)^\Delta + \mu\left(\frac{\eta_1}{\eta}\right)^\Delta + \frac{2\eta_1}{\eta} - 2 + \frac{b\mu}{\eta^\sigma}$$
$$= (b\mu^2-a)\left(\frac{1}{\eta}\right)^\Delta + \mu\left(\frac{\eta_1}{\eta}\right)^\Delta + \frac{2\eta_1}{\eta} - 2 + \frac{b\mu}{\eta} = 0$$

we get

$$\left(\frac{\eta_1}{\eta}\right)^{\Delta} = -\frac{2}{\mu}\frac{\eta_1}{\eta} + \frac{a-b\mu^2}{\mu}\left(\frac{1}{\eta}\right)^{\Delta} + \frac{2}{\mu} - \frac{b}{\eta}$$

with a solution (5.87).

Further, in view of $1/\eta^{\sigma} = 1/\eta + \mu(1/\eta)^{\Delta}$, we have

$$\begin{aligned}\frac{a^{\sigma}HF}{\eta^{\sigma}} &= -a\left(\frac{1}{\eta}\right)^{\Delta} + \frac{\eta_1^{\sigma}}{\eta^{\sigma}} + \frac{\eta_1}{\eta} - 2 + \frac{b\mu}{\eta^{\sigma}}\\ &= (b\mu^2 + \eta_1^{\sigma}\mu - a)\left(\frac{1}{\eta}\right)^{\Delta} + \frac{\eta_1^{\sigma} + \eta_1 + b\mu}{\eta} - 2\end{aligned}$$

and $HF = 0$ is satisfied if

$$\left(\frac{1}{\eta}\right)^{\Delta} = \frac{\Omega}{\eta} - \frac{2}{a - \eta_1^{\sigma}\mu - b\mu^2}$$

with a solution (5.88). □

Proof of Theorem 4.1. We deduce Theorem 4.1 from Theorem 2.2 applied to the system

$$\begin{pmatrix}\psi(t)\\ \psi^{\Delta}(t)\end{pmatrix}^{\Delta} = \begin{pmatrix}0 & 1\\ -\frac{b(t)}{a^{\sigma}(t)} & \frac{a^{\Delta}(t)+b(t)\mu(t)}{a^{\sigma}(t)}\end{pmatrix}\begin{pmatrix}\psi(t)\\ \psi^{\Delta}(t)\end{pmatrix},$$

which is equivalent to equation (4.1), with a characteristic function (4.4). Since

$$a_{12} = 1, \qquad CL_k(t) = \frac{CF_k(t)}{a^{\sigma}(t)a(t)},$$

$$\frac{e_2(t)}{e_1(t)} = \frac{e_{\eta_2/a}(t)}{e_{\eta_1/a}(t)} = e_r(t), \qquad r(t) = \frac{\eta_2(t) - \eta_1(t)}{a(t) + \mu(t)\eta_1(t)},$$

we get from (2.13) condition (4.6), and from formula (2.15) we have

$$\begin{aligned}M_1(t) = M &= \max_{k,j=1,2}\left|\frac{e_j CL_j(t)}{\theta^{\sigma}e_k^{\sigma}}\right|\\ &= \max_{k,j=1,2}\left|\frac{a^{\sigma}e_j CL_j}{\eta^{\sigma}(1+\mu\eta_k/a)e_k}\right|\\ &= \max_{k,j=1,2}\left|\frac{e_r^{\pm 1}CF_j}{\eta^{\sigma}(a+\mu\eta_k)}\right|.\end{aligned}$$

□

Proof of Theorem 4.2. Theorem 4.2 is deduced from Theorem 4.1 by choosing approximation

$$\eta_1 = 0, \tag{5.90}$$

then, from Lemma 5.3, we get $CF_1 = CF_2$ if we choose

$$\begin{aligned} \eta_2(t) &= \frac{1}{\int_{t_1}^{t} \frac{e_{\Omega_1}(t,\sigma(s))\Delta s}{a-b\mu^2}} = \frac{1}{\int_{t_1}^{t} \frac{e_{\Omega_1}(t,s)\Delta s}{a(s)}}, \\ \Omega_1(t) &= \frac{b(t)\mu(t)}{a(t)-b(t)\mu^2(t)}. \end{aligned} \tag{5.91}$$

By this choice, regressivity condition (4.5) follows from (4.11). Further,

$$CF_1(t) = CF_2(t) = a(t)b(t), \tag{5.92}$$

$$\widetilde{e_1} = 1, \quad \widetilde{e_2}(t) = e_r(t), \quad r(t) = \frac{\eta_2(t)}{a(t)} = \frac{1}{a(t)\int_{t_1}^{t}(2e_{\Omega_1}(t,s)\Delta s)/a(s)}. \tag{5.93}$$

Since $\eta(t) = (\eta_1 - \eta_2)/2 = -(\eta_2)/2$ condition (4.6) of Theorem 4.1 becomes

$$\left|\frac{\widetilde{e_1(t)}\widetilde{e_2(t)}\eta(t)}{a(t)}\right| = \frac{r(t)e_r(t)}{2} \neq 0, \quad t > t_0,$$

and is satisfied.

$$\begin{aligned} M_2 = M &= \max\left|\frac{e_r^{\pm 1}CF_1}{\eta^\sigma(a+\mu\eta_k}\right| = \max\left|\frac{e_r^{\pm 1}ab}{a^\sigma r^\sigma(a+\mu a r\delta_{k2})}\right| \\ &= \max\left|\frac{e_r^{\pm 1}(t)b(t)}{a^\sigma(t)r^\sigma(t)}\right|. \end{aligned}$$ □

Proof of Remark 4.1. By definition of $r(t)$ condition (4.14) means

$$0 \le r(t) \le \frac{1}{t-t_0},$$

and (4.15) follows from the monotonicity of $e_r(t,t_0)$. □

Proof of Example 1.1. Consider the more general Cauchy problem for the one-dimensional wave equation for $(t,x) \in \mathbf{T}_0 \times R_x$

$$u^{\Delta\Delta}(t,x) = (a^2 + b(t))u_{xx}(t,x) + c(t)u_x, \qquad (5.94)$$
$$u(t_0,x) = f(x), \qquad u^{\Delta}(t_0,x) = 0,$$

where Δ is a delta derivative by the time variable.

Using Fourier transformation by the x variable from (5.94) we get an ordinary dynamic equation on a time scale

$$\widehat{u}^{\Delta\Delta}(t,\xi) + (\xi^2 a^2 + \xi^2 b(t) + i\xi c(t))\widehat{u}(t,\xi) = 0, \qquad (5.95)$$
$$\widehat{u}(t_0,\xi) = \widehat{f}(\xi), \quad \widehat{u}^{\Delta}(t_0,\xi) = 0.$$

Let us check that, under assumption,

$$\int_{t_0}^{\infty} \frac{|b(t)|\Delta t}{\mu(t)} < \infty, \qquad \int_{t_0}^{\infty} |c(t)|\Delta t < \infty, \qquad (5.96)$$

the conditions of Theorem 3.3 are satisfied. Choosing $\theta_{1,2} = \pm im$, $m = a\xi$ condition (3.6) is satisfied if $\xi \neq 0$. To check condition (3.7), note that, from $m = a\xi \neq 0$, we have $q = (-2im\mu)/(1+im\mu)$, $1 + q\mu = (1-im\mu)/(1+im\mu)$, $|1+q\mu| = 1$ and

$$|e_q(t,t_0)| = 1.$$

Indeed, if $\mu \equiv 0$ this is trivial. If $\mu(t) > 0$, then it follows from definition $|e_q(t,t_0)| = \exp\int_{t_0}^{t}(\mathrm{Log}\,|1+q\mu|\Delta s)/\mu(s) = 1$.

Further,

$$R(t) = \xi^2 a^2 + \xi^2 b(t) + i\xi c(t), \quad \theta_1 = ia\xi,$$
$$R - m^2 = b\xi^2 + i\xi c(t)$$

$$M(t) = \left|\frac{e_q^{\pm 1}(R-m^2)}{m(1+\mu\theta_1)}\right| = \frac{|b\xi^2 + i\xi c(t)|}{a|\xi|\sqrt{1+\mu^2\xi^2}}$$
$$\le \frac{|b\xi^2|}{a|\xi|\sqrt{1+\mu^2\xi^2}} + \frac{|i\xi c(t)|}{a|\xi|\sqrt{1+\mu^2\xi^2}}$$

or

$$M(t) \le \frac{|b(t)|}{a\mu(t)} + \frac{|c(t)|}{a} \in L_1(t_0,\infty).$$

Introducing trigonometric functions on a time scale

$$\cos_\xi(t,t_0)=\frac{e_{i\xi}(t,t_0)+e_{-i\xi}(t,t_0)}{2},$$
$$\sin_\xi(t,t_0)=\frac{e_{i\xi}(t,t_0)-e_{-i\xi}(t,t_0)}{2i},$$

from Theorem 3.3, the solutions of (5.94) may be written in the asymptotic form:

$$\widehat{u}(t,\xi)=[C_1(\xi)+\varepsilon_1(t,\xi)]\sin_{a\xi}(t,t_0)+[C_2(\xi)+\varepsilon_2(t,\xi)]\cos_{a\xi}(t,t_0),$$
$$\widehat{u}^\Delta(t,\xi)=a\xi[C_1(\xi)+\varepsilon_1(t,\xi)]\cos_{a\xi}(t,t_0)+a\xi[C_2(\xi)+\varepsilon_2(t,\xi)]\sin_{a\xi}(t,t_0)$$

with the error estimate

$$\|\varepsilon(t,\xi)]\|\le\|C\|(e^{\int_t^\infty M(s)\Delta s}-1).$$

Further, from initial conditions (assuming $\varepsilon_j(t_0,\xi)=0$), we get

$$C_1(\xi)+\varepsilon_1(t_0,\xi)=C_1(\xi)\equiv 0,\qquad C_2(\xi)+\varepsilon_2(t_0,\xi)=C_2(\xi)=\widehat{f}(\xi),$$

so

$$\widehat{u}(t,\xi)=[\widehat{f}(\xi)+\varepsilon_2(t,\xi)]\cos_{a\xi}(t,t_0)\in S'(R_\xi).$$

Using inverse Fourier transformation, we get time scale analogue of D'Alembert's formula:

$$u(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ix\xi}[\widehat{f}(\xi)+\varepsilon_2(t,\xi)]\cos_{a\xi}(t,t_0)\,d\xi,\qquad (5.97)$$
$$\lim_{t\to\infty}\varepsilon_2(t,\xi)=0.$$

By using the formula for the Fourier transform for $\widehat{f}(\xi)$ we can simplify (5.97) as:

(5.98)
$$u(t,x)=\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{i\xi(y-x)}\left(e^{\int_{t_0}^t\frac{\mathrm{Log}\,(1+ia\xi\mu)\Delta s}{\mu}}+e^{\int_{t_0}^t\frac{\mathrm{Log}\,(1-ia\xi\mu)\Delta s}{\mu}}\right)$$
$$\times[f(y)+\delta_2(t,y)]\,dy\,d\xi.$$

Assuming that $|a\xi\mu(t)|\le 1/2$ from the Taylor series

$$a\xi\frac{\mathrm{Log}\,(1+ia\mu\xi)}{a\mu\xi}=a\xi\left(i+\frac{a\xi\mu}{2}-\frac{ia^2\xi^2\mu^2}{3}+\cdots\right)$$

we get a solution as a sum of two waves moving in opposite directions:

$$(5.99) \qquad \begin{aligned} u(t,x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} & e^{i\xi(y-x)+ia\xi(t-t_0)} \\ & \times [f(y)+\delta_2(t,y)] e^{\int_{t_0}^{t} \left(\frac{1}{2}a^2\xi^2\mu - \frac{i}{3}a^3\xi^3\mu^2 + \cdots\right)\Delta s} dy\, d\xi \\ & + \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{i\xi(y-x)-ia\xi(t-t_0)} \\ & \times [f(y)+\delta_2(t,y)] e^{\int_{t_0}^{t} \left(\frac{1}{2}a^2\xi^2\mu - \frac{i}{3}a^3\xi^3\mu^2 + \cdots\right)\Delta s} dy\, d\xi. \end{aligned}$$

In the case $\xi\mu = 0$, $\delta_2(t,y) = 0$, we get the classical D'Alembert's formula

$$(5.100) \qquad u(t,x) = \frac{f(x+a(t-t_0)) + f(x-a(t-t_0))}{2}.$$

□

REFERENCES

1. R. Aulbach and S. Hilger, *Linear dynamic processes with inhomogeneous time scale*, in *Non-linear dynamics and quantum dynamical systems*, Acad. Verlag, Berlin, 1990.

2. Z. Benzaid and D.A. Lutz, *Asymptotic representation of solutions of perturbed systems of linear difference equations*, Stud. Appl. Math. **77** (1987), 195–221.

3. G.D. Birkhoff, *General theory of linear difference equations*, Trans. Amer. Soc. **12** (1911), 243–284.

4. S. Bodine, M. Bohner and D.A. Lutz, *Asymptotic behavior of solutions of dynamic equations*, J. Math. Sci. **124** (2004), 5110–5118.

5. M. Bohner and D. Lutz, *Asymptotic behavior of dynamic equations on time scales*, J. Diff. Equat. Appl. **7** (2001), 21–50.

6. ———, *Asymptotic expansion and analytic time scales*, ZAMM-Z. Math. Mech. **86** (2006), 37–45.

7. M. Bohner and A. Peterson, *Dynamic equations on time scales*: *An introduction with applications*, Birkhauser, Boston, 2001.

8. ———, *Advances in dynamic equations on time scales*, Birkhauser, Boston, 2002.

9. M. Bohner and S. Stevic, *Asymptotic behavior of second-order dynamic equations*, Appl. Math. Comp. **188** (2007), 1503–1512 .

10. L. Cesary, *Asymptotic behavior and stability problems in ordinary differential equations*, 3rd edition, Springer Verlag, Berlin, 1970.

11. C.V. Coffman, *Asymptotic behavior of solutions of ordinary difference equations*, Trans. Amer. Math. Soc. **110** (1964), 22–51.

12. M.S.P. Eastham, *The asymptotic solution of linear differential systems. Applications of the Levinson theorem*, Clarendon Press, Oxford, 1989.

13. T. Gard and J. Hoffacker, *Asymptotic Behavior of natural growth on time scales*, Dyn. Syst. Appl. **12** (2003), 131–148.

14. W.A. Harris and D.A. Lutz, *On the asymptotic integration of linear differential systems*, J. Math. Anal. Appl. **48** (1974), 1–16.

15. ———, *A unified theory of asymptotic integration*, J. Math. Anal. Appl. **57** (1977), 571–586.

16. P. Hartman, *Ordinary differential equations*, John Wiley and Sons, New York, 1973.

17. S. Hilger, *Analysis on measure chains–a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.

18. G. Hovhannisyan, *Levinson theorem for* 2×2 *system and applications to the asymptotic stability and Schrodinger equation*, Inter. J. Evol. Equat. **3** (2008), 181-203.

19. ———, *Asymptotic stability for dynamic equations on time scales*, Adv. Diff. Equat. Article ID **18157** (2006), 17 pages.

20. ———, *Asymptotic stability for* 2×2 *linear dynamic systems on time scales*, Inter. J. Diff. Equat. **2** (2007), 105–121.

21. ———, *On oscillations of solutions of* n*-th order dynamic equation*, ISRN Math. Anal. **2013** (2013), Article ID 946453, 11 pages.

22. G. Hovhannisyan and W. Liu, *On non-autonomous Dirac equation*, J. Math. Phys. **50** (2009), 123507, 24 pages.

23. N. Levinson, *The asymptotic nature of solutions of linear systems of differential equations*, Duke Math. J. **15** (1948), 111–126.

24. C. Potzsche, S. Siegmund and F. Wirth, *A spectral characterization of exponential stability for linear time-unvariant systems on time scales*, Discr. Cont. Dyn. Syst. **9** (2003), 1223–1241.

25. G. Ren, Y. Shi and Y. Wang, *Asymptotic behavior of solutions of perturbed linear difference systems*, Linear Alg. Appl. **395** (2005), 283–302.

26. V. Volterra, *Sui fondamenti della teoria delle equazioni differenziali lineari*, Mem. Soc. Ital. Sci. **6** (1887), 1–104.

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