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Asymptotic behavior of n-th order dynamic equations

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Abstract: We are concerned with the asymptotic behavior of solutions of an n-th order linear dynamic equation on a time scale in terms of Taylor monomials. In particular, we describe the asymptotic behavior of the so-called (first) principal solution in terms of the Taylor monomial of degree $n-1$. Several interesting properties of the Taylor monomials are established so that we can prove our main results.

Keywords: *asymptotic behavior; dynamic equations; time scale; Taylor monomials, oscillation.*

Mathematics Subject Classification (2000): 34E10, 39A10

1 Introduction

We shall first consider the two term n-th order linear dynamic equation

$$u^{\Delta^n} + p(t)u(t) = 0, \quad p(t) > 0, \quad t \geq t_0 \tag{1}$$

on a time scale $\mathbb{T}$. Later (see Theorem 2.4) we consider a more general n-th order linear dynamic equation with $n+1$ terms. For the sake of completeness, we recall some basic definitions from the theory of time scales [7, 14].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Since we are interested in oscillation results, we will consider time scales which are unbounded above , i.e., $\sup(\mathbb{T}) = \infty$. We use the notation $\mathbb{T} := [t_0, \infty)$.

For $t \in \mathbb{T}$ we define the forward and backward jump operators

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T}, s < t\}. \tag{2}$$

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The (forward) graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t. \tag{3}$$

If $\mathbb{T}$ has a left-scattered minimum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$ define the delta derivative $f^\Delta(t)$ to be the number (provided it exists) with the property that for any $\epsilon > 0$, there exists a $\delta > 0$ and a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ of t such that

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|, \quad f^\sigma(t) \equiv f(\sigma(t)), \tag{4}$$

for all $s \in U$, (see [7]). A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and at each left-dense point t in $\mathbb{T}$ the left hand limit at t exists (finite). The set of rd-continuous functions on $\mathbb{T}$ will be denoted by C_{rd}. The set of functions such that their n-th delta derivative exists and is rd-continuous on $\mathbb{T}$ is denoted by C^n_{rd}. In (1) we assume that $p \in C_{rd}$ and we say x is a solution provided $x \in C^n_{rd}$ and $u^{\Delta^n}(t) + p(t)u(t) = 0$ for $t \in \mathbb{T}^\kappa$. We say that a function f is regressive on $\mathbb{T}$ if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}$. The set of regressive functions on $\mathbb{T}$ which belong to C_{rd} is denoted by $\mathcal{R}$. The set of regressive functions in C^n_{rd} will be denoted by $\mathcal{R}^n$.

A solution u of (1) is said to have a zero at $a \in \mathbb{T}$ if $u(a) = 0$, and it has a generalized zero at a if either $u(t)$ has a zero at a or if $u(\rho(a))u(a) < 0$ A solution of (1) is said to be oscillatory if it has an infinite sequence of generalized zeros in $\mathbb{T}$, and nonoscillatory otherwise. Equation (1) is said to be **oscillatory** if all solutions are oscillatory and is said to be **nonoscillatory** if all solutions are nonoscillatory. An interesting question is what conditions guarantee the existence of both (i.e, coexistence). Oscillation theorems for n-th order differential equations have been established by many authors. One often finds criteria under which all solutions are oscillatory. The approach here is somewhat different in that we are interested in establishing sufficient conditions for the existence of at least one oscillatory solution or conditions which guarantee that all solutions are nonoscillatory with a certain asymptotic form. We refer to the results of W. Leighton and Z. Nehari [21], I. M. Glazman [12], G. V. Anan'eva and V. I. Balaganskii [2], V. A. Kondrat'ev [17], I. T. Kiguradze [16], the book of Swanson [24], and the many references therein.

Oscillation theorems for second order dynamic equations on a time scale have been studied by many authors since the introduction of the time scale calculus by Hilger [14]. As examples, we refer to the results in [11, 4, 18]. In this paper we establish some sufficient conditions for the existence of an oscillatory solution and for nonoscillation of the n-th order equation (1) on a time scale in terms of the Taylor monomials. We also mention that some oscillation results for (1) were obtained in [20]. For additional related results on the asymptotic behavior of solutions of dynamic equations see [6, 15, 23, 25].

2 Main Results

We recall the definition of the Taylor monomials (these Taylor monomials were first introduced by Agarwal and Bohner in [1]) as follows:

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau, \quad k = 0, 1, 2, 3, \cdots, \quad h_0(t, s) = 1, \quad t \ge s. \tag{5}$$

The solution $u = u(\cdot, t_1)$ of the IVP (1),

$$u(t_1) = u^\Delta(t_1) = \cdots = u^{\Delta^{n-2}}(t_1) = 0, \quad u^{\Delta^{n-1}}(t_1) = 1, \quad t_1 > t_0. \tag{6}$$

is called the **principal solution** of (1) at t_1.

Our first result gives a 'smallness' condition (7) on an integral involving the Taylor monomials which guarantees that the principal solution is nonoscillatory.

Theorem 2.1 *If $p \in C_{rd}$, and*

$$\int_{t_0}^{\infty} h_{n-1}(s,t_0)p(s)\Delta s < \infty, \tag{7}$$

then the principal solution u of (1) is eventually positive. Moreover, (7) holds if and only if

$$\lim_{t\to\infty} \frac{u(t)}{h_{n-1}(t,t_1)} = C > 0. \tag{8}$$

Theorem 2.2 *If $p \in C_{rd}$, and u is a solution of (1) which is eventually positive, then*

$$\lim_{t\to\infty} \frac{u(t)}{h_{n-1}(t,t_1)} = \lim_{t\to\infty} u^{\Delta^{n-1}}(t) := L, \tag{9}$$

where $0 < L < +\infty$. That is, both limits are finite and positive.

Theorem 2.3 *If $p \in C_{rd}$, and*

$$\int_{t_1}^{\infty} h_{n-2}(t,t_1)p(t)\Delta t = \infty, \tag{10}$$

then equation (1) has at least one oscillatory solution.

Remark 2.1 *If*

$$\lim_{t\to\infty} \frac{\mu(t)}{t} = 0, \tag{11}$$

then Theorem 2.3 is true if, instead of (10), the simpler condition

$$\int_{t_1}^{\infty} t^{n-2}p(t)\Delta t = \infty \tag{12}$$

is satisfied. More generally, if for some number $K \in (0,1)$,

$$\frac{\mu(t)}{t} \le (n-1)^{\frac{1}{n-2}}\left(K^{\frac{1}{2-n}} - 1\right), \quad n \ge 4, \tag{13}$$

and (12) are satisfied then the conclusion of Theorem 2.3 is true. In general, however (12) does not imply (10) as is shown in the following example.

Example 2.1 *Consider the time scale $\mathbb{T}_1 = \{t_k = 2^{2^k}, k = 0,1,2,3,\cdots\}$ (see [7]). For this time scale there are functions p such that*

$$\int_1^{\infty} h_2(t,t_1)p(t)\Delta t < \infty$$

but

$$\int_1^{\infty} t^2 p(t)\Delta t = \infty.$$

The proof of this example is given at the end of Section 3.

Using the asymptotic representation method [22, 10, 9] one can prove the following theorem.

Theorem 2.4 *Assume that for all $j = 1, \cdots, n$, we have $p_j \in C_{rd}$, and*

$$\int_{t_0}^{\infty} |p_j(t)| h_{[j/2]}(t,t_0) h_{j-1-[j/2]}(t,t_0) \frac{h_{j-1}^{\sigma}(t,t_0)}{h_{j-1}(t,t_0)} \left(\frac{h_1^{n-1}(t,t_0)}{h_{n-1}(t,t_0)} \right)^{\sigma} \Delta t < \infty, \tag{14}$$

where $[j/2]$ is the integral part of $\frac{j}{2}$. Then the equation

$$u^{\Delta^n} + p_1(t) u^{\Delta^{n-1}} + \cdots + p_{n-1}(t) u^{\Delta}(t) + p_n(t) u(t) = 0, \quad t \in \mathbb{T} \tag{15}$$

is nonoscillatory on $\mathbb{T} \cap [t_1, \infty)$.

Remark 2.2 *If* (13) *is true, then equation* (15) *is nonoscillatory if the simpler condition*

$$\int_t^{\infty} \sigma^{j-1}(s) |p_j(s)| \Delta s < \infty, \quad j = 1, \cdots, n. \tag{16}$$

is satisfied.

Note that under assumption (16), the asymptotic behavior of solutions of (15) on a continuous time scale ($\sigma(s) = s$) was described by Ghizzetti [12].

Remark 2.3 *When $n = 3$, equation* (1) *is nonoscillatory if*

$$\int_{t_1}^{\infty} \frac{t^2 \sigma^2(t) p(t) \Delta t}{h_2(t,t_1)} < \infty, \tag{17}$$

and it has at least one oscillatory solution if

$$\int_{t_1}^{\infty} t p(t) \Delta t = \infty. \tag{18}$$

Before beginning the proofs, we would like to mention some consequences for the n-th order linear difference equation

$$\Delta^n x(k) + p(k) x(k) = 0, \tag{19}$$

where $p(k) \geq 0$. It was shown in [20] that all solutions are oscillatory in case

$$\sum_1^{\infty} k^{n-1-\epsilon} p(k) = \infty, \tag{20}$$

for some $0 < \epsilon < n-1$ when n is even, and every solution is either oscillatory or $\lim_{n\to\infty} x(n) = 0$ when n is odd. However, when $\epsilon = 0$, the result is no longer valid. The results in the present paper show that if $\sum_1^{\infty} k^{n-2} p(k) = \infty$, then there exists at least one oscillatory solution. If $\sum_1^{\infty} k^{n-1} p(k) < \infty$, then the equation is nonoscillatory.

3 Proofs

In the proof of the main results we use the methods developed in [21]. We shall need various estimates on the Taylor monomials which we collect in the following lemma.

Lemma 3.1 *The Taylor monomials satisfy the following properties:*

$$h_n(t_1,s) \geq h_n(t_2,s), \quad t_1 \geq t_2 \geq s, \quad h_n(t,s_1) \leq h_n(t,s_2), \quad t \geq s_1 \geq s_2, \tag{21}$$

$$h_n(t,t_1) \geq h_{n-1}(t,t_1), \quad \lim_{t\to\infty} h_n(t,t_1) = \infty, \quad t \geq t_1+1, \quad n = 1,2,\cdots, \tag{22}$$

$$\lim_{t\to\infty} \frac{h_k(t,t_1)}{h_{n-1}(t,t_1)} = 0, \quad k = 0,1,\cdots,n-2, \quad \lim_{t\to\infty} \frac{h_k(t,t_2)}{h_k(t,t_1)} = 1, \tag{23}$$

$$(t-s)^n \leq \frac{((t-s)^{n+1})^{\Delta}}{n+1}, \quad \int_s^t (\tau-s)^n \Delta\tau \leq \frac{(t-s)^{n+1}}{n+1}, \quad n = 0,1,\cdots \quad t \geq s, \tag{24}$$

$$h_n(t,s) \leq \frac{(t-s)^n}{n!} = \frac{h_1^n(t,s)}{n!}, \quad n = 0,1,\cdots, \quad t \geq s > 0, \tag{25}$$

$$\frac{h_{k-1}(t,s)}{h_k^{\sigma}(t,s)} \leq \frac{h_k(t,s)}{h_{k+1}^{\sigma}(t,s)}, \quad h_{q-1}(t,s)h_{j-q}(t,s) \leq h_q(t,s)h_{j-q-1}(t,s), \quad t \geq s, \tag{26}$$

where $1 \leq k \leq n, \quad 1 \leq q \leq j/2.$

Suppose that for some positive integer m there exists a number $A \in (0,1)$ such that

$$\frac{\mu(t)}{t} \leq S_m, \quad S_m = (m+1)^{\frac{1}{m}}(A^{-\frac{1}{m}} - 1), \quad t > 0. \tag{27}$$

Then

$$S_{k+1} < S_k, \quad k = 1,2,\cdots, \tag{28}$$

and

$$t^k \geq A\frac{(t^{k+1})^{\Delta}}{k+1}, \quad k = 1,2,\cdots,m. \tag{29}$$

If (27) *is true for $m = n$, then*

$$h_n(t,s) \geq B_{n-1}t^n - (1 + B_1 + 2!B_2 + \cdots + (n-1)!B_{n-1})\frac{t^{n-1}s}{(n-1)!}, \tag{30}$$

where

$$B_n = \frac{A^n}{(n+1)!}, \quad B_0 = 1, \quad n = 0,1,2,\cdots. \tag{31}$$

Proof The statement concerning the monotone increasing nature of $h_n(t,s)$ in the first argument is trivial. We prove the monotone decreasing property of $h_n(t,s)$ in the second argument by induction. That is, we will show

$$h_{n-1}(t,s_1) \leq h_{n-1}(t,s_2), \quad s_1 \geq s_2, \quad n = 1,2,\cdots.$$

If $n = 1$ the statement is trivial. Assuming that the result is true for $n-1$, we see that (21) holds for n since

$$h_n(t,s_1) = \int_{s_1}^{t} h_{n-1}(\tau,s_1)\Delta\tau \le \int_{s_1}^{t} h_{n-1}(\tau,s_2)\Delta\tau \le \int_{s_2}^{t} h_{n-1}(\tau,s_2)\Delta\tau = h_n(t,s_2).$$

We also establish property (22) by induction. For $n = 1$, (22) follows from the formula $h_1(t,t_1) = t - t_1 \ge 1$. Assuming that (22) is true for $n = 1, 2, \cdots, k$, we obtain

$$h_{k+1}(t,t_1) = \int_{t_1}^{t} h_k(\tau,t_1)\Delta\tau \ge \int_{t_1}^{t} h_{k-1}(\tau,t_1)\Delta\tau = h_k(t,t_1),$$

which completes the induction.

From these inequalities we get

$$h_n(t,t_1) \ge h_1(t,t_1) = t - t_1, \quad n \ge 1,$$

and the property $\lim_{t\to\infty} h_n(t,t_1) = \infty$.
To prove (23) we will use L'Hospital's rule:

Lemma 3.2 *[7] Assume f and g are differentiable on $\mathbb{T}$ with*

$$\lim_{t\to\infty} g(t) = \infty,$$

$$g(t) > 0, \quad g^{\Delta}(t) > 0, \quad t \in \mathbb{T}.$$

Then

$$\lim_{t\to\infty} \frac{f^{\Delta}(t)}{g^{\Delta}(t)} = r$$

implies

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = r.$$

Indeed, since $h_n^{\Delta}(t,t_1) = h_{n-1}(t,t_1)$, then using (22) we have

$$\lim_{t\to\infty} \frac{h_1(t,t_1)}{h_2(t,t_1)} = \lim_{t\to\infty} \frac{h_0(t,t_1)}{h_1(t,t_1)} = \lim_{t\to\infty} \frac{1}{t-t_1} = 0.$$

The general case of (23) is proved similarly.

To prove (24) we note that

$$((t-s)^{n+1})^{\Delta} = \sum_{k=0}^{n} (\sigma(t)-s)^k (t-s)^{n-k} \ge \sum_{k=0}^{n} (t-s)^k (t-s)^{n-k} = (n+1)(t-s)^n.$$

The second inequality in (24) is proved by integration of the previous inequality.

Inequality (25) may again be established by induction (see also [6, Theorem4.1] for a proof of this result). For $n = 0$ it is clear. Assuming

$$h_{n-1}(t,s) \le \frac{(t-s)^{n-1}}{(n-1)!}$$

we have

$$h_n(t,s) = \int_s^t h_{n-1}(\tau,s)\Delta\tau \le \int_s^t \frac{(\tau-s)^{n-1}}{(n-1)!}\Delta\tau \le \int_s^t \frac{((\tau-s)^n)^\Delta}{n!}\Delta\tau = \frac{(t-s)^n}{n!}.$$

To prove the first inequality (26) it is enough to prove that

$$\frac{h_{k-1}(t,s)}{h_k(t,s)} \le \frac{h_k(t,s)}{h_{k+1}(t,s)}, \quad k = 1,2,\cdots \tag{32}$$

in view of

$$\frac{h_k(t,s)}{h^\sigma_{k+1}(t,s)} - \frac{h_{k-1}(t,s)}{h^\sigma_k(t,s)} = \frac{h^2_k(t,s) - h_{k+1}(t,s)h_{k-1}(t,s)}{h^\sigma_{k+1}(t,s)h^\sigma_k(t,s)}.$$

We will prove (32) by induction, and that the sequence $\frac{h_{k-1}(t,s)}{h_k(t,s)}$, $k = 1,2,\cdots$ is decreasing with respect to t.

For $k=1$ we have the sequence $\frac{1}{h_1(t,s)} = \frac{1}{t-s}$ is decreasing with respect to t, and

$$\frac{h_0(t,s)}{h_1(t,s)} \le \frac{h_1(t,s)}{h_2(t,s)}$$

which follows from (25): $h_2(t,s) \le \frac{h_1^2(t,s)}{2}$.

Assuming that $\frac{h_{k-1}(t,s)}{h_k(t,s)}$ is decreasing with respect to t and (32) is true for k we have

$$\left(\frac{h_k(t,s)}{h_{k+1}(t,s)}\right)^\Delta = \frac{h_{k-1}(t,s)h_{k+1}(t,s) - h^2_k(t,s)}{h^\sigma_{k+1}(t,s)h_{k+1}(t,s)} \le 0.$$

That is, $\frac{h_k(t,s)}{h_{k+1}(t,s)}$ is decreasing with respect to t, and

$$h_{k+2}(t,s) = \int_s^t \frac{h_{k+1}(\tau,s)}{h_k(\tau,s)} h_k(\tau,s)\Delta\tau \le$$

$$\frac{h_{k+1}(t,s)}{h_k(t,s)} \int_s^t h_k(\tau,s)\Delta\tau = \frac{h_{k+1}(t,s)h_{k+1}(t,s)}{h_k(t,s)}$$

which gives (32) with $k \to k+1$:

$$\frac{h_k(t,s)}{h_{k+1}(t,s)} \le \frac{h_{k+1}(t,s)}{h_{k+2}(t,s)}.$$

The second inequality (26) may be proved by using the property of Taylor monomials that the ratio $\frac{h_{j-q-1}(t,s)}{h_{q-1}(t,s)}$ is increasing in t if $j-q-1 \ge q-1$, or $q \le j/2$. Indeed

$$h_{j-q}(t,s) = \int_{t_0}^t \frac{h_{j-q-1}(z,s)}{h_{q-1}(z,s)} h_{q-1}(z,s)\Delta z \le \frac{h_{j-q-1}(t,s)}{h_{q-1}(t,s)} \int_{t_0}^t h_{q-1}(z,s)\Delta z =$$

$$\frac{h_{j-q-1}(t,s)}{h_{q-1}(t,s)} h_q(t,s).$$

To prove (28) note that both sequences $(k+1)^{1/k}$ and $A^{-1/k}$ are decreasing with respect to k for $k \ge 1$.

First we prove (29) for the case $k = m$. Since

$$(t^{m+1})^{\Delta} = \sum_{k=0}^{m} \sigma^k(t) t^{m-k} = t^m + \sigma(t)t^{m-1} + \cdots + \sigma^{m-1}(t)t + \sigma^m(t),$$

to prove (28) with $k = m$ from (27) it is enough to prove

$$t^m \geq \frac{A}{m+1} \sum_{k=0}^{m} \sigma^k(t) t^{m-k}.$$

If $\mu \equiv 0$, it is trivial with $A = 1$. Assuming $\mu \neq 0$ and dividing the inequality by t^m we have

$$1 \geq \frac{A}{m+1} \sum_{k=0}^{m} (x+1)^k, \quad \text{where} \quad x = \frac{\mu(t)}{t},$$

so that summing the right hand side gives

$$1 \geq \frac{A}{m+1} \frac{(x+1)^{m+1} - 1}{x} = \frac{A}{m+1}(x^m + C_{m+1}^1 x^{m-1} + \cdots + C_{m+1}^{m-2} x^2 + C_{m+1}^{m-1} x + C_{m+1}^m),$$

where C_{m+1}^k is the binomial coefficient. Hence the inequality holds if

$$1 \geq A \left(\frac{x^m}{m+1} + \frac{C_{m+1}^1 x^{m-1}}{m+1} + \cdots + \frac{C_{m+1}^{m-1} x}{m+1} + 1 \right).$$

Now this inequality is true if

$$1 \geq A \left(\frac{x}{(m+1)^{\frac{1}{m}}} + 1 \right)^m \tag{33}$$

$$= A \left(\frac{x^m}{m+1} + \frac{C_m^1 x^{m-1}}{(m+1)^{\frac{m-1}{m}}} + \frac{C_m^2 x^{m-2}}{(m+1)^{\frac{m-2}{m}}} + \frac{C_m^3 x^{m-3}}{(m+1)^{\frac{m-3}{m}}} + \cdots + 1 \right)$$

is satisfied, since

$$\frac{C_m^k x^{m-k}}{(m+1)^{\frac{m-k}{k}}} \geq \frac{C_{m+1}^k x^{m-k}}{m+1},$$

or

$$(m+1-k)^m \geq (m+1)^{m-k}, \quad k = 0, 1, \cdots, m, \quad m = 0, 1, 2, \cdots.$$

To see this note that if $m = 0, \quad k = 0$, it is true, and if it is true for $m, \quad k = 0, 1, \cdots, m$ then it is true for $m \to m+1, \quad k = 0, 1, \cdots, m+1$. Now we need to show that

$$(m+2-k)^{m+1} \geq (m+2)^{m+1-k}, \quad k = 0, 1, \cdots, m+1.$$

To prove this, we do another induction on k: If $k = 0$, it is true. Assuming

$$(m+2)^{m+1-k} \leq (m+2-k)^{m+1}$$

is true, we obtain the result for $k \to k+1$ as follows:

$$(m+2)^{m-k} = \frac{(m+2)^{m+1-k}}{m+2} \leq \frac{(m+2-k)^{m+1}}{m+2} \leq (m+3-k)^{m+1},$$

or dividing by $(m+2-k)^{m+1}$ we get

$$\frac{1}{m+2} \leq \left(\frac{m+3-k}{m+2-k}\right)^{m+1}.$$

which is true, since the left side is less than one, and the right side is greater than 1. Furthermore from (33) we have

$$1 \geq A^{\frac{1}{m}} \left(\frac{x}{(m+1)^{\frac{1}{m}}} + 1\right),$$

so by assumption (27) we have

$$\frac{t}{\mu(t)} = \frac{1}{x} \geq \frac{A^{\frac{1}{m}}}{(m+1)^{\frac{1}{m}}(1-A^{\frac{1}{m}})}.$$

To prove (29) for all $k = 1, \cdots, m$, note that if (27) is true for $m > 1$, then it is true for all $k = 1, 2, \cdots, m$ since the sequence S_k is decreasing.

The last property (30)

$$h_n(t,s) \geq B_{n-1}t^n - (1 + B_1 + 2!B_2 + \cdots + (n-1)!B_{n-1})\frac{t^{n-1}s}{(n-1)!},$$

we prove again by induction. When $n = 1$, it is obvious. Assuming (29) is true we prove it for $n \to n+1$. From (27)

$$\begin{aligned} h_{n+1}(t,s) &= \int_s^t h_n(\tau,s)\Delta\tau \\ &\geq \int_s^t \left(B_{n-1}\tau^n - (1 + B_1 + 2!B_2 + \cdots + (n-1)!B_{n-1})\frac{t^{n-1}s}{(n-1)!}\right)\Delta\tau \\ &\geq \frac{A_n}{n+1}B_{n-1}(t^{n+1} - s^{n+1}) - \frac{1}{n!}(1 + B_1 + 2!B_2 + \cdots + (n-1)!B_{n-1})(t^n - s^n)s \\ &\geq B_n t^{n+1} - B_n t^n s - \frac{1}{n!}(1 + B_1 + 2!B_2 + \cdots + (n-1)!B_{n-1})t^n s \end{aligned}$$

in view of $t^{n-1} \leq \frac{(t^n)^\Delta}{n}$. That is,

$$h_{n+1}(t,s) \geq B_n t^{n+1} - (1 + B_1 + 2!B_2 + \cdots + (n-1)!B_{n-1} + n!B_n)\frac{t^n s}{n!}.$$

which completes the proof.

Proof of Theorem 2.1

Since (7) holds we may take t_1 large enough so that

$$\int_t^\infty h_{n-1}(s,t_1)p(s)\Delta s \leq \frac{1}{2}, \quad t \geq t_1. \tag{34}$$

Assume the principal solution of (1) is oscillatory. Then there are two possibilities:

1. There exists a point $t_2 \in \mathbb{T}, t_2 > t_1$, where the principal solution has a zero: $u(t_2, t_1) = 0$ and $u(t, t_1) > 0$ on (t_1, t_2).

2. There exists a point $t_2 \in \mathbb{T}, t_2 > t_1$, where $u(t_2, t_1) > 0$ and $u(\sigma(t_2), t_1) < 0$.

In the first case, from Taylor's formula

$$u(t) = u(t_1) + u^{\Delta}(t_1)(t - t_1) + u^{\Delta\Delta}(t_1)h_2(t, t_1) + \cdots + u^{\Delta^{n-1}}(t_1)h_{n-1}(t, t_1)+$$
$$\int_{t_1}^{t} u^{\Delta^n}(s)h_{n-1}(t, \sigma(s))\Delta s, \quad h_k^{\Delta}(t, s) = h_{k-1}(t, s), \quad k = 0, 1, 2, \cdots, n-1,$$

so that the principal solution of equation (1) can be written in the form

$$u(t, t_1) = h_{n-1}(t, t_1) - \int_{t_1}^{t} h_{n-1}(t, \sigma(s))p(s)u(s, t_1)\Delta s. \tag{35}$$

From (35) we have

$$u(t, t_1) \le h_{n-1}(t, t_1),$$
$$h_{n-1}(t_2, t_1) = \int_{t_1}^{t_2} h_{n-1}(t_2, \sigma(s))p(s)u(s, t_1)\Delta s \le h_{n-1}(t_2, \sigma(t_1)) \int_{t_1}^{t_2} p(s)u(s, t_1)\Delta s \le$$
$$h_{n-1}(t_2, \sigma(t_1)) \int_{t_1}^{t_2} p(s)h_{n-1}(s, t_1)\Delta s.$$

Dividing this inequality by $h_{n-1}(t_2, \sigma(t_1))$ we get

$$\int_{t_1}^{t_2} h_{n-1}(s, t_1)p(s)\Delta s \ge \frac{h_{n-1}(t_2, t_1)}{h_{n-1}(t_2, \sigma(t_1))}.$$

Using the monotonicity of the Taylor monomial with respect to its second argument, we get

$$\int_{t_1}^{t_2} h_{n-1}(s, t_1)p(s)\Delta s \ge 1,$$

which contradicts (34).

In the second case, from (35)

$$u(\sigma(t_2), t_1) = h_{n-1}(\sigma(t_2), t_1) - \int_{t_1}^{\sigma(t_2)} h_{n-1}(\sigma(t_2), \sigma(s))p(s)u(s, t_1)\Delta s < 0$$

$$h_{n-1}(\sigma(t_2), t_1) < \int_{t_1}^{\sigma(t_2)} h_{n-1}(\sigma(t_2), \sigma(s))p(s)u(s, t_1)\Delta s \le$$

$$h_{n-1}(\sigma(t_2), \sigma(t_1)) \int_{t_1}^{\sigma(t_2)} p(s)u(s, t_1)\Delta s \le h_{n-1}(\sigma(t_2), \sigma(t_1)) \int_{t_1}^{\sigma(t_2)} h_{n-1}(s, t_1)p(s)\Delta s$$
$$\int_{t_1}^{\sigma(t_2)} h_{n-1}(s, t_1)p(s)\Delta s \ge \frac{h_{n-1}(\sigma(t_2), t_1)}{h_{n-1}(\sigma(t_2), \sigma(t_1))} \ge 1,$$

or

$$\int_{t_1}^{\sigma(\infty)} h_{n-1}(s, t_1)p(s)\Delta s \ge \int_{t_1}^{\sigma(t_2)} h_{n-1}(s, t_1)p(s)\Delta s \ge 1,$$

so we get a contradiction again. Therefore, we conclude that the principal solution is nonoscillatory.
From (35)

$$\begin{aligned} h_{n-1}(t,t_1) &= u(t,t_1) + \int_{t_1}^{t} h_{n-1}(t,\sigma(s))p(s)u(s,t_1)\Delta s \\ &\le u(t,t_1) + h_{n-1}(t,\sigma(t_1)) \int_{t_1}^{t} p(s)u(s,t_1)\Delta s \\ &\le u(t,t_1) + h_{n-1}(t,t_1) \int_{t_1}^{t} p(s)h_{n-1}(s,t_1)\Delta s \\ &\le u(t,t_1) + \frac{1}{2}h_{n-1}(t,t_1), \end{aligned}$$

so we get the inequality

$$\frac{1}{2}h_{n-1}(t,t_1) \le u(t,t_1) \le h_{n-1}(t,t_1). \tag{36}$$

Before proving the second part of Theorem 2.1, we prove Theorem 2.2.

Proof of Theorem 2.2.
Assuming that $u(t)$ is a positive solution of (1) on $[t_1,\infty)$, we have from Taylor's formula and (1) that

$$R(t) = u(t) + \int_{t_1}^{t} h_{n-1}(t,\sigma(s))p(s)u(s)\Delta s, \tag{37}$$

where $R(t)$ is the polynomial

$$R(t) = \sum_{k=0}^{n-1} h_k(t,t_1)u^{\Delta^k}(t_1), \tag{38}$$

or

$$R(t) = u(t_1) + (t-t_1)u^{\Delta}(t_1) + h_2(t,t_1)u^{\Delta\Delta}(t_1) + \cdots + h_{n-1}(t,t_1)u^{\Delta^{n-1}}(t_1).$$

Since $u > 0, \quad t \ge t_1$, and $h_{n-1}(t,s)$ is decreasing in the second argument, we have from (37)

$$\begin{aligned} R(t) &\le u(t) + h_{n-1}(t,t_1) \int_{t_1}^{t} p(s)u(s)\Delta s \\ &= u(t) + h_{n-1}(t,t_1)\left(u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(t)\right), \end{aligned}$$

where (1) has been used in the last step. Dividing by $h_{n-1}(t,t_1)$ we get

$$\frac{R(t)}{h_{n-1}(t,t_1)} \le \frac{u(t)}{h_{n-1}(t,t_1)} + u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(t).$$

In view of $\lim_{t\to\infty} \frac{R(t)}{h_{n-1}(t,t_1)} = u^{\Delta^{n-1}}(t_1)$, if t tends to infinity through a sequence of points for which $\frac{u(t)}{h_{n-1}(t,t_1)}$ approaches its lower limit we have

$$\lim_{t\to\infty} u^{\Delta^{n-1}}(t) \le \liminf_{t\to\infty} \frac{u(t)}{h_{n-1}(t,t_1)}. \tag{39}$$

Note that the limit $u^{\Delta^{n-1}}(t)$ as $t \to \infty$ exists since, by (1), $u^{\Delta^{n-1}}(t)$ decreases. Choosing $t_1 < \xi < t$ from (37) we have

$$\begin{aligned} R(t) &\geq u(t) + \int_{t_1}^{\xi} h_{n-1}(t, \sigma(s)) p(s) u(s) \Delta s \\ &\geq u(t) + h_{n-1}(t, \sigma(\xi)) \int_{t_1}^{\xi} p(s) u(s) \Delta s \\ &= u(t) + h_{n-1}(t, \sigma(\xi)) \left(u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(\xi) \right). \end{aligned}$$

From

$$\frac{R(t)}{h_{n-1}(t,t_1)} \geq \frac{u(t)}{h_{n-1}(t,t_1)} + \frac{h_{n-1}(t,\sigma(\xi))}{h_{n-1}(t,t_1)} \left(u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(\xi) \right)$$

we get

$$\frac{h_{n-1}(t,\sigma(\xi)) u^{\Delta^{n-1}}(\xi)}{h_{n-1}(t,t_1)} + \frac{R(t)}{h_{n-1}(t,t_1)} \geq \frac{u(t)}{h_{n-1}(t,t_1)} + \frac{h_{n-1}(t,\sigma(\xi)) u^{\Delta^{n-1}}(t_1)}{h_{n-1}(t,t_1)}$$

or

$$u^{\Delta^{n-1}}(\xi) + \frac{R(t)}{h_{n-1}(t,t_1)} \geq \frac{u(t)}{h_{n-1}(t,t_1)} + \frac{h_{n-1}(t,\sigma(\xi)) u^{\Delta^{n-1}}(t_1)}{h_{n-1}(t,t_1)}.$$

since

$$\frac{h_{n-1}(t,\sigma(\xi))}{h_{n-1}(t,t_1)} \leq 1.$$

Now as $t \to \infty$ using (23) we get

$$u^{\Delta^{n-1}}(\xi) + u^{\Delta^{n-1}}(t_1) \geq \limsup_{t\to\infty} \frac{u(t)}{h_{n-1}(t,t_1)} + \limsup_{t\to\infty} \frac{h_{n-1}(t,\sigma(\xi)) u^{\Delta^{n-1}}(t_1)}{h_{n-1}(t,t_1)}$$

and since from (23)

$$\lim_{t\to\infty} \frac{h_{n-1}(t,\sigma(\xi))}{h_{n-1}(t,t_1)} = 1$$

we have

$$u^{\Delta^{n-1}}(\xi) + u^{\Delta^{n-1}}(t_1) \geq \limsup_{t\to\infty} \frac{u(t)}{h_{n-1}(t,t_1)} + u^{\Delta^{n-1}}(t_1)$$

or

$$\limsup_{t\to\infty} \frac{u(t)}{h_{n-1}(t,t_1)} \leq \lim_{\xi\to\infty} u^{\Delta^{n-1}}(\xi), \tag{40}$$

which with (39) proves Theorem 2.2.

Returning to the proof of Theorem 2.1, recall that under assumption (7) it was shown that the principal solution $u(t,t_1)$ satisfies inequalities (36).

By Theorem 2.2

$$\lim_{t\to\infty} \frac{u(t)}{h_{n-1}(t,t_1)}$$

exists and it is positive. So condition (7) is sufficient for the existence of a solution with the prescribed asymptotic behavior.

To prove the necessity, we assume that (1) has a solution such that

$$\lim_{t\to\infty} \frac{u(t)}{h_{n-1}(t,t_1)} = c > 0. \tag{41}$$

Evidently this assumption ensures that $u(t)$ is ultimately nonoscillatory, otherwise the limit in question would be zero. Now we may assume that $u(t)$ is positive, and from Theorem 2.2 $\lim_{t\to\infty} u^{\Delta^{n-1}}(t) = c$. Integrating (1), we get that

$$\int_{t_1}^{\infty} p(t)u(t)\Delta t = u^{\Delta^{n-1}}(t_1) - c.$$

From our assumption, $u(t) \geq (c-\varepsilon)h_{n-1}(t,t_1)$ for some $\varepsilon > 0$, and so

$$u^{\Delta^{n-1}}(t_1) - c \geq (c-\varepsilon)\int_{t_1}^{\infty} t^{n-1}p(t)h_{n-1}(t,t_1)\Delta t,$$

and it follows that $\int_{t_1}^{\infty} h_{n-1}(t,t_1)p(t)\Delta t < \infty$. This completes the proof of Theorem 2.1.

Proof of Theorem 2.3.

Assume that (10) holds but equation (1) is nonoscillatory on (t_1,∞). Then the principal solution $u(t,t_1)$ will be positive for $t > t_1$. From (1) $u^{\Delta^n}(t,t_1) < 0$, $u^{\Delta^{n-1}}(t,t_1)$ is decreasing. By Theorem 2.2 $\lim_{t\to\infty} u^{\Delta^{n-1}}(t,t_1)$ is positive, so $u^{\Delta^{n-1}}(t,t_1) > 0, \quad t > t_1$ and

$$u^{\Delta^{n-2}}(t,t_1) = \int_{t_1}^{t} u^{\Delta^{n-1}}(s)\Delta s > A > 0, \quad t > t_1,$$

and since $u(b,t_1) > 0$ if b is slightly larger than t_1, we have

$$u(t) = u(b) + \cdots + h_{n-3}(t,b)u^{\Delta^{n-3}}(b) + \int_b^t h_{n-3}(t,\sigma(s))u^{\Delta^{n-2}}(s)\Delta s$$

$$u(t,t_1) \geq u(b) + h_1(t,b)u^{\Delta}(b) + ... + h_{n-3}(t,b)u^{\Delta^{n-2}}(b) + Ah_{n-2}(t,b) > Ah_{n-2}(t,b).$$

On the other hand

$$u^{\Delta^{n-1}}(b,t_1) = u^{\Delta^{n-1}}(t,t_1) + \int_b^t p(t)u(t,t_1)\Delta t > \int_b^t p(t)u(t,t_1)\Delta t > A\int_b^t p(t)h_{n-2}(t,b)\Delta t,$$

and when $t \to \infty$ we get

$$\int_b^t p(t)h_{n-2}(t,b)\Delta t < \infty,$$

which contradicts (10).

Proof of Remark 2.1.

If condition (13) is satisfied, then (27) is true for $m = n-2$, and from (30) we have for some small positive $\varepsilon > 0$

$$h_{n-2}(t,t_1) \geq t^{n-2}(B_{n-3} - \varepsilon)$$

which implies Remark 2.1.

Proof of Example 2.1.

For the time scale $\mathbb{T}_1 = \{t_k = 2^{2^k}, k = 0, 1, 2, 3, \cdots\}$ we have

$$\sigma(t) = t^2, \quad \mu(t) = t^2 - t, \quad h_1(t) = h_1(t, 2) = t - 2$$

and for $m \geq 1$

$$h_2(t_m, 2) = \sum_{k=0}^{m-1} h_1(t_k)\mu(t_k) = \sum_{k=0}^{m-1}(t_k - 2)(t_k^2 - t_k) \leq \sum_{k=0}^{m-1} t_k(t_k^2 - t_k) \leq t_{m-1}^3 = t_m^{3/2},$$

where we used the inequality

$$\sum_{k=0}^{m-1}(t_k^3 - t_k^2) \leq t_{m-1}^3, \quad m \geq 1,$$

which may be proved by induction. To see this note that it is true for $m = 1$, and if it is true for m, then it is true for $m \to m + 1$ as well:

$$\sum_{k=0}^{m}(t_k^3 - t_k^2) = \sum_{k=0}^{m-1}(t_k^3 - t_k^2) + t_m^3 - t_m^2 \leq t_{m-1}^3 + t_m^3 - t_m^2 \leq t_m^3.$$

Further choosing $p(t) = t^{-4-\varepsilon_k}, \quad \varepsilon_k = 2^{-k} = \frac{1}{\log_2(t_k)} > 0$ we have

$$\int_1^\infty h_2(t)p(t)\Delta t = \sum_{k=1}^{\infty} h_2(t_k)p(t_k)\mu(t_k) \leq \sum_{k=1}^{\infty} t_k^{3/2} t_k^{-4-\varepsilon}(t_k^2 - t_k) \leq$$

$$\sum_{k=1}^{\infty} t_k^{-1/2-\varepsilon} = \sum_{k=1}^{\infty} \frac{1}{2 \cdot 2^{2^{k-1}}} < \infty.$$

However

$$\int_1^\infty t^2 p(t)\Delta t = \sum_{k=1}^{\infty} t_k^2 p(t_k)\mu(t_k) =$$

$$\sum_{k=1}^{\infty} t_k^2 t_k^{-4-\varepsilon}(t_k^2 - t_k) \geq \frac{1}{2}\sum_{k=1}^{\infty} t_k^{-\varepsilon} = \frac{1}{2}\sum_{k=1}^{\infty} 2^{-1} = \infty.$$

This establishes the validity of Example 2.1.

4 Proof of Theorem 2.4

To prove Theorem 2.4 we will construct explicit nonoscillating asymptotic solutions of (15). Since different asymptotic methods ([22, 10, 4]) are used in the proof of Theorem 2.4, we include the proof of it in this special case.

The equation

$$u^{\Delta^n} + p_1(t)u^{(\Delta)^{n-1}} + \cdots + p_{n-1}(t)u^\Delta(t) + p_n(t)u(t) = 0, \quad t \in \mathbb{T} \tag{42}$$

may be written as a system

$$x^\Delta(t) = (J + P(t))x(t) \tag{43}$$

where

$$x(t) = \begin{pmatrix} u^{(\Delta)^{n-1}} \\ \dots \\ u^{\Delta} \\ u \end{pmatrix}, \quad P(t) = \begin{pmatrix} -p_1 & -p_2 & \dots & -p_n \\ 0 & 0 & \dots & 0 \\ . & . & . & \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & \dots & \dots & 1 & 0 \end{pmatrix}.$$

Using the transformation

$$x(t) = \Lambda(t) y(t), \tag{44}$$

where

$$\Lambda(t) = e^J D(t), \quad D(t) = \operatorname{diag}\{h_0, h_1, \cdots, h_{n-1}\},$$

that is

$$\Lambda(t) = \begin{pmatrix} h_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ h_1 & h_1 & 0 & 0 & \dots & 0 & 0 \\ h_2 & h_1^2 & h_2 & 0 & \dots & 0 & 0 \\ h_3 & h_2 h_1 & h_1 h_2 & h_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ h_{n-1} & h_{n-1} h_1 & \dots & \dots & \dots & \dots & h_{n-1} \end{pmatrix},$$

(here we suppress the dependence on t: $h_j = h_j(t, t_0), \quad p_j = p_j(t)$) we get

$$y^{\Delta}(t) = (E(t) + B(t)) y(t), \tag{45}$$

where by direct calculations

$$E(t) = (\Lambda^{-1})^{\sigma} \left(J\Lambda(t) - \Lambda^{\Delta}(t) \right) = -(D^{\sigma})^{-1} D^{\Delta}(t), \quad B(t) = (\Lambda^{-1})^{\sigma} P \Lambda. \tag{46}$$

Here

$$E(t) = \operatorname{diag}\{\theta_1(t), \cdots, \theta_n(t)\}, \tag{47}$$

where

$$\theta_1(t) = 0, \quad \theta_k(t) = -\frac{h_{k-2}(t, t_0)}{h_{k-1}(\sigma(t), t_0)}, \quad k = 2, 3, \cdots, n.$$

From property (26) the sequence $\theta_k, \quad k = 1, 2, \cdots$, is decreasing with respect to k, that is,

$$\theta_{k+1}(t) < \theta_k(t), \quad k = 1, 2, \cdots, n-1, \quad t \geq t_1 > t_0.$$

Note that $\theta_k \in \mathcal{R}$ since $1 + \mu\theta_k = \frac{h_{k-1}(t, t_0)}{h_{k-1}(\sigma(t), t_0)} > 0$. Consider the solutions of the n^2 initial value problems

$$w_{ij}^{\Delta}(t) = q(t) w_{ij}(t), \quad w_{ij}(t_1) = 1, \quad q(t) = \frac{\theta_j(t) - \theta_i(t)}{1 + \mu(t)\theta_i(t)}. \tag{48}$$

Note that solutions of (48) exist and are unique, if $q \in \mathcal{R}$.

To find asymptotic representations of solutions of (45) we will apply a time scale version of Levinson's theorem (for further results on the time scale version of Levinson's Theorem see [3]):

Theorem 4.1 *[4] Assume that* $\theta_k \in \mathbb{R}, \quad 1 \le k \le n$

$$\int_{t_1}^{\infty} \left| \frac{B(t)\Delta t}{1+\mu(t)\theta_j(t)} \right| < \infty, \quad 1 \le j \le n, \tag{49}$$

and suppose that there exists a number $m > 0$ *such that for each pair* (i,j) *with* $i \ne j$, *solutions* $w_{ij}(t)$ *of* (48) *satisfy*

$$\lim_{t\to\infty} w_{ij}(t) = 0, \quad \left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| \ge m, \quad t_1 \le s \le t. \tag{50}$$

Then the linear system (45) *has a fundamental matrix* $Y(t)$ *such that*

$$Y(t) = [I + \varepsilon(t)]V(t), \quad \lim_{t\to\infty} \varepsilon(t) = 0, \tag{51}$$

where $\varepsilon(t)$ *is the error matrix-function, and* $V(t)$ *satisfies*

$$V^{\Delta}(t) = E(t)V(t), \quad V(t_1) = I. \tag{52}$$

Since the matrix E is diagonal (see (47)), and $\theta_j \in \mathcal{R}$, one can find solutions of (52) in terms of the Euler exponential functions:

$$v_j(t) = e_{\theta_j}(t, t_1), \quad j = 1, 2, 3, \cdots, n,$$

or in terms of the Taylor monomials:

$$e_{\theta_j}(t, t_1) = \frac{h_{j-1}(t_1, t_0)}{h_{j-1}(t, t_0)}, \quad j = 1, \cdots, n, \quad t \ge t_1 > t_0. \tag{53}$$

Note that $v_j(t) = e_{\theta_j}(t, t_1), \quad j = 1, 2, \cdots, n$ are nonoscillatory. Solutions of (48) are

$$w_{ij}(t) = e_q(t, t_1) = \frac{e_{\theta_j}(t, t_1)}{e_{\theta_i}(t, t_1)}, \quad q = \frac{\theta_j - \theta_i}{1 + \mu\theta_i}, \quad j > i.$$

Since $\theta_j < \theta_i, \quad j > i$, we have $q(t) < 0$, but $q \in \mathcal{R}$, in view of $1 + q\mu = \frac{1+\mu\theta_j}{1+\mu\theta_i} < 0$. From (48), (53) we get

$$w_{ij}(t) = \frac{h_{j-1}(t_1, t_0)h_{i-1}(t, t_0)}{h_{j-1}(t, t_0)h_{i-1}(t_1, t_0)}, \quad t \ge t_1 > t_0. \tag{54}$$

Before applying Theorem 4.1 let us check the conditions. From (23), condition (50) is satisfied:

$$\lim_{t\to\infty} w_{ij}(t) = \lim_{t\to\infty} \frac{h_{j-1}(t_1, t_0)h_{i-1}(t, t_0)}{h_{j-1}(t, t_0)h_{i-1}(t_1, t_0)} = 0, \quad j > i \ge 1,$$

$$\left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| = \frac{h_{i-1}(s, t_0)h_{j-1}(t, t_0)}{h_{j-1}(s, t_0)h_i(t, t_0)} \ge \frac{h_{i-1}(s, t_0)h_{i-1}(t, t_0)}{h_{i-1}(s, t_0)h_{i-1}(t, t_0)} = 1, \quad j > i \ge 1.$$

To check condition (49) note that by direct calculations from (46)

$$B_{n,k} = h_{k-1} \sum_{j=k}^{n} p_j h_{j-k} \left(\frac{h_1^{n-1} - (n-2)h_1^{n-3}h_2 - \cdots}{h_{n-1}} \right)^{\sigma}.$$

In view of (25),(26) we have

$$h_{k-1}h_{j-k} \le h_{[j/2]}h_{j-1-[j/2]} \quad 1 \le k \le j,$$

$$\left|\frac{h_1^{n-1} - (n-2)h_1^{n-3}h_2 - \cdots}{h_{n-1}}\right| \le C\frac{h_1^{n-1}}{h_{n-1}},$$

so

$$h_{k-1}\sum_{j=k}^{n}|p_j|h_{j-k} \le \sum_{j=1}^{n}|p_j|h_{[j/2]}h_{j-1-[j/2]}, \quad \text{for all} \quad 1 \le k \le j,$$

and

$$|B_{nk}| \le C\sum_{j=1}^{n}|p_j|h_{[j/2]}h_{j-1-[j/2]}\left(\frac{h_1^{n-1}}{h_{n-1}}\right)^{\sigma}.$$

Therefore

$$\|B(t)\| = C\sum_{j=1}^{n}|p_j|h_{[j/2]}h_{j-1-[j/2]}\left(\frac{h_1^{n-1}}{h_{n-1}}\right)^{\sigma}. \tag{55}$$

So condition (49) becomes

$$\int_{t_0}^{\infty}|p_j(t)|h_{[j/2]}(t,t_0)h_{j-1-[j/2]}(t,t_0)\frac{h_{j-1}^{\sigma}(t,t_0)}{h_{j-1}(t,t_0)}\left(\frac{h_1^{n-1}(t,t_0)}{h_{n-1}(t,t_0)}\right)^{\sigma}\Delta t < \infty, \quad j = 1,\cdots,n,$$

which is condition (14).

From Theorem 4.1 we get the asymptotic representation (51)

$$Y(t) = \Lambda(t)(I + \varepsilon(t))]V(t).$$

The fundamental matrix solution X of (43) in view of (44) may be written in the form

$$X(t) = \Lambda(t)Y(t) = \Lambda(t)(I + \varepsilon(t))]V(t),$$

and solutions u of equation (42) are not oscillatory.

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