ASYMPTOTIC SOLUTIONS OF SINGULAR HYPERBOLIC EQUATIONS

Gro R. Hovhannisyan

Izvestiya Natsionalnoi Akademii Nauk Armenii. Matematika, Vol. 31, No. 3, 1996

Abstract

The paper establishes local asymptotic representations for solutions of linear singular hyperbolic equations by means of Fourier integral operators. It is assumed that the coefficients of the equations are unbounded near a singular hyperplane $t = 0$. These representations generalize the well known Levinson's asymptotic theorem from the theory of ordinary differential equations. They are useful for the study of some equations of mathematical physics. Another application is in the study of correctness of Cauchy problem for the partial differential equations with multiple characteristics.

§1. INTRODUCTION

It is well known, that solutions of strictly hyperbolic equations with smooth coefficients can be represented by means of Fourier integral operators. For example, the equation of oscillating string

$$v_{tt} = a^2 v_{xx}, \tag{1.1}$$

using Fourier transformation

$$\hat{v}(t,\xi) = \int v(t,x) \exp(-ix\xi)\, dx$$

is replaced by the ordinary linear differential equation

$$\hat{v}_{tt} + a^2\xi^2\hat{v} = 0, \tag{1.2}$$

which has the general solution

$$\hat{v}(t,\xi) = \hat{f}(\xi)\exp\{iat \mid \xi \mid\} + \hat{g}(\xi)\exp\{-iat \mid \xi \mid\}$$

with arbitrary functions $\hat{f}(\xi)$ and $\hat{g}(\xi)$. Using inverse Fourier transformation

$$v(t,x) = (2\pi)^{-n}\int \hat{v}(t,\xi)\exp(ix\xi)\,d\xi,$$

we get

$$v(t,x) = \Phi_1(t)f(y) + \Phi_2(t)g(y), \tag{1.3}$$

where the operators $\Phi_1(t)$ and $\Phi_2(t)$ have integral representations

$$\Phi_j f(y) = (2\pi)^{-n}\int f(y)\exp(i\theta_j(t,x,y,\xi)\,dy\,d\xi, \quad j = 1,2, \tag{1.4}$$

$$\theta_1 = (x-y)\xi + at|\xi|, \quad \theta_2 = (x-y)\xi - at|\xi|.$$

Here the functions f and g are from Sobolev spaces. For the second order partial differential equation

$$Lu = u_{tt} - a^2u_{xx} + q(t,x)u = 0, \quad t \in]0,T[, \quad x \in R, \quad \lim_{t\to 0} q = \infty \tag{1.5}$$

the following question can be posed: under what restrictions on operators $L\Phi_j$, $j = 1,2$ solutions of (1.5) can be represented in the form

$$u(t,x) = \hat{\Phi}_1(t)f(y) + \hat{\Phi}_2(t)g(y), \tag{1.6}$$

where

$$\hat{\Phi}_j(t) = \Phi_j(t) + \varepsilon_j(t,x), \quad \varepsilon_j(t,x) \to 0, \quad t \to 0, \quad j = 1,2. \tag{1.6$'$}$$

$\Phi_j(t)$ are called *asymptotic resolvent operators*, $\varepsilon_j(t,x)$ are called *error functions.*

For more general strictly hyperbolic (or hyperbolic in Petrovsky's sense) equations the solutions $u(t,x)$ admit representations (see [2], [7])

$$u(t,x) = \sum_{j=1}^{m} \hat{\Phi}_j(t) C_j(y), \tag{1.7}$$

where $C_j(y)$ are arbitrary functions from Sobolev spaces and $\hat{\Phi}_j$ are linear integral operators, that can be represented as sums of Fourier integral operators. Below we prove the following asymptotic versions of the representations (1.7) for the solutions of a wide class of t-hyperbolic (non-strictly hyperbolic, weakly hyperbolic) operators with coefficients unbounded on the initial hyperplane $t = 0$:

$$u(t,x) = \sum_{j=1}^{m} [\Phi_j(t) + \varepsilon_j(t,y)] g_j(y), \quad \lim_{t \to 0} \varepsilon_j(t,y) = 0, \quad j = 1, \ldots, m. \tag{1.8}$$

We apply (1.8) in the study initial value problems near singular hyperplane.

§2. THE MAIN RESULTS

This section contains formulations of the theorems and propositions, proved in the paper. Theorem 2.1 provides a sufficient criterion of asymptotic resolvent operators. Theorem 2.2 is an existence and uniqueness theorem for weighted initial value problem, generalizing the initial value problem with Wronskian data considered in [10]. Propositions 2.1, 2.3, 2.4 are particular cases of theorem 2.2 for second order partial differential equations with simplified initial data. In Propositions 2.2, 2.5 we propose a construction of linear integral operators that occur in asymptotic representations (2.13). Note that Proposition 2.2 is a generalization of the JWKB asymptotics well known in the theory of ordinary differential equations for hyperbolic partial differential equations.

We consider the partial differential equation

$$Pu = \sum_{k=0}^{m} Q_{m-k}(t,x,D_x) \partial_t^k u(t,x) = 0, \quad t \in [0,T], \quad x \in R^n, \tag{2.1}$$

where

$$Q_0 = 1, \quad Q_j \in C^\infty(]0,T], \Psi^j_\rho), \quad j = 1, ..., m, \tag{2.2}$$

and Ψ^j_ρ is a class of pseudodifferential operators to be defined below. By B we denote the set of all C^∞-functions, defined and bounded in R^n_x with derivatives of all orders. Let $B^k_t = C^k([0,T], B)$. We say that a C^∞-function $p(x,y,\xi)$ in $R^{3n} = R^n_x \times R^n_y \times R^n_\xi$ belongs to $S^m_{\rho,\delta}$ $(0 \leq \delta < 1/2 < \rho \leq 1)$, if for any multi-indices α, β, γ we have

$$| \, \partial^\alpha_\xi \partial^\beta_x \partial^\gamma_y p(x,y,\xi)| \leq C_{\alpha,\beta,\gamma} < \xi >^{m+\delta|\beta+\gamma|-\rho|\alpha|} \quad \text{on } R^{3n}, \tag{2.3}$$

where $< \xi >= (1 + \xi_1^2 + \cdots + \xi_n^2)^{1/2}$. As usual $S^m_{\rho,\delta}$ are Frechet spaces provided with seminorms

$$|p|^{(m)}_l = \max_{\alpha+\beta+\gamma|\leq l} \inf\{C_{\alpha,\beta,\gamma} \quad \text{of (2.3)}\}. \tag{2.4}$$

We say that a symbol $p(x,\xi)$ of class $S^m_\rho = S^m_{\rho,1-\rho}$ belongs to the class $S^m_\rho((k))$, where $k \geq 0$ is an integer, if

$$p^{(\alpha)}_{(\beta)}(x,\xi) \in S^{m-|\alpha|)}_\rho \quad (| \, \alpha + \beta \, | \leq k).$$

We say that real-valued C^∞-function $I(x,\xi)$ in $R^n_x \times R^n_\xi$ belongs to the class $\mathcal{P}_\rho(\tau)$, where $0 \leq \tau \leq 1$ and $1/2 \leq \rho \leq 1$ (see [4]), if $J(x,\xi) = \varphi(x,\xi) - x\xi$ satisfies

$$J(x,\xi) \in S^1_\rho((2)),$$

$$||J||_0 = \sum_{|\alpha+\beta\leq 2} \sup_{x,\xi}\{|J^{(\alpha)}_{(\beta)}(x,\xi)| < \xi >^{(|\alpha|-1)} \leq \tau.$$

In particular, for $\rho = 1$ we write $\mathcal{P}(\tau)$. For any integer $l \geq 1$, we define the subclass $\mathcal{P}(\tau, l)$ of $\mathcal{P}_\rho(\tau)$ to be the set of phase functions $I(x,\xi) \in \mathcal{P}_\rho(\tau)$ satisfying

$$J(x,\xi) \in S^1_\rho((2)),$$

$$||J||_l = \sum_{|\alpha+\beta|\le 1} \sup_{x,\xi}\{|J^{(\alpha)}_{(\beta)}(x,\xi)| < \xi >^{(|\alpha|-1)} +$$

$$+ \sum_{2\le|\alpha+\beta|\le 2+l} \sup_{x,\xi}\{|J^{(\alpha)}_{(\beta)}(x,\xi)| < \xi >^{1-(\rho-1)|\beta|+\rho(|\alpha|-2)}\} \le \tau,$$

and write $\mathcal{P}(\tau,l)$ for $\mathcal{P}_1(\tau,l)$ for $\rho=1$. Let $\varphi(x,\xi)\in\mathcal{P}_\rho(\tau)$. For $p(x,\xi)\in S^m_\rho$ we define the Fourier integral operator P with phase function $\varphi(x,\xi)$ and symbol $p(x,\xi)$ by

$$Pu(x) = P_I u(x) = (2\pi)^{-2n}\int e^{i(I(x,\xi)-y\xi)}p(x,\xi)u(y)\,d\xi\,dy, \quad \text{for } u\in B \quad (2.5)$$

in the sense of regularized oscillatory integrals ([4]). We write $P\in I^m_\rho$ and $\sigma(P)=p(x,\xi)$. For $\rho=1$ we shall often write I^m for I^m_ρ.

Let $\Psi^m_{\rho,\delta}$ be the class of pseudodifferential operators with symbols from $S^m_{\rho,\delta}$ ($\Psi^m_\rho=\Psi^m_{\rho,1-\rho}$, $\Psi^m=\Psi^m_{1,0}$), $\widetilde{I}^m$ be the class of linear integral operators, mapping H^s to H^{s-m}. Recall that an operator (2.1) is called *t-hyperbolic (or weakly, non strictly hyperbolic)*, if the roots $\tau=\lambda_j(t,x,\xi)$, $j=1,...,m$, of the characteristic equation

$$\sum_{k=0}^{m} q_{m-k}(t,x,\xi)\tau^k = 0, \quad q_0 = 1$$

are real-valued for every $t\in]0,T[$, $(x,\xi)\in R^{2n}$. Here $q_{m-k}(t,x,\xi)$ are the principal symbols of the operators Q_{m-k}. We suppose that the functions q_{m-k}, $k=0,1,\ldots,m$ are real-valued for every $t\in]0,T]$, $x\in R^n$, $\xi\in R^n$.

Let the operators $\{\Phi_j(t)\}_{j=1}^m$ from (1.7) be given. Instead of the operators $\Phi_j\in\tilde{I}^{1-j}$ we can construct auxiliary linear integral operators

$$\Phi_{s+1,s}(t), \quad \Phi^{-1}_{s+1,s}(t)\in\tilde{I}^0, \quad s=0,\ldots m-1. \quad (2.8)$$

Indeed, we define

$$\Phi_{j,s}(t) = \Phi_{s+1,s}(t)Z_{j,s+1}(t), \quad j=s+2,\ldots m, \quad s=0,\ldots,m-2, \quad (2.9)$$

$$\Phi_{j,0} = \Phi_j, \quad j = 1, \dots, m, \tag{2.9$'$}$$

$$Z_{j,s+1}(t) = \int_T^t \Phi_{j,s+1}(\tau)\, d\tau, \quad j = s+2, .., m, \quad s = 0, \dots, m-2. \tag{2.9$''$}$$

From (2.9) we get

$$\Phi_{s+2,s} = \Phi_{s+1,s} \int_T^t \Phi_{s+2,s+1}(\tau)\, d\tau, \quad s = 0, \dots m-1,$$

and $\Phi_{s+1,s}(t)$ can be determined from these expressions recurrently. In particular,

$$\Phi_1(t) = \Phi_{1,0}(t), \quad \Phi_2(t) = \Phi_{1,0}(t) \int_T^t \Phi_{2,1}(\tau)\, d\tau,$$

$$\Phi_{3,0}(t) = \Phi_{1,0}(t) \int_T^t \left(\Phi_{2,1}(\tau) \int_T^\tau \Phi_{3,2}(s)\, ds \right) d\tau. \tag{2.10}$$

We consider the operators

$$K_j^s = \{ - (\Phi_{2,1}\Phi_{3,2} \cdots \Phi_{m,m-1})^{-1} P \Phi_j, \text{if} \quad j = 1, \dots, m, \quad s = m,$$

$$K_j^s = \sum (-1)^k Z_{j_1,s} Z_{j_2,j_1} \cdots Z_{j_k,j_{k-1}} K_j^m, \quad \text{if} \quad j = 1, \dots, m, \quad s < m. \tag{2.11}$$

where summations are over all indexes $j_1, \dots, j_k$, such that

$$s < j_1 < j_2 < \dots < j_k = m, \quad k = 1, \dots, m-s, \quad s = 1, \dots, m-1. \tag{2.11$'$}$$

Theorem 2.1 Let the functions $\beta_s(t) \geq 0$, $\beta_s \in L_1[0,T]$, $s = 1, \dots, m$ and infinitely differentiable by parameter $t \in]0,T]$, invertible linear integral operators $\{\Phi_j(t)\}_{j=1}^m$, satisfy the conditions (2.8) and

$$(1/\beta_s(t)) K_j^s(t) \in \tilde{I}^0, \quad s, j = 1, \dots, m, \tag{2.12}$$

uniformly in $t \in]0,T[$. Then for any choice of functions $C_j(y) \in B$, $j = 1, \dots, m$, exists a solution $u \in B_t^\infty$ of the equation (2.1) representable in the form:

$$\partial_t^{k-1} u = \sum_{j=1}^m (\partial_t^{k-1} \Phi_j)[C_j(y) + \varepsilon_j(t,y)], \quad k = 1, \dots, m, \tag{2.13}$$

$$\lim_{t\to 0} ||\varepsilon_j(t,.)||_p = 0, \quad j = 1,\dots,m, \text{ for some } p \in R. \tag{2.13'}$$

Remark 2.1. Theorem 2.1 gives no construction of the operators Φ_j, but it is useful, because one can choose approximate operators Φ_j in different ways, depending on the equation.

Remark 2.2. Let P_2 be the operator (2.1) in case $m = 2$. In this case the conditions $(2.8), (2.12)$ of Theorem 2.1 become

$$\Phi_1, \Phi_1^{-1} \in \tilde{I}^0, \tag{2.14}$$

$$\Phi_{2,1}, \Phi_{2,1}^{-1} \in \tilde{I}^0, \tag{2.15}$$

$$\beta_s^{-1}(t)\Phi_j^{-1}(\Phi_{1t}\Phi_1^{-1} - \Phi_{2t}\Phi_2^{-1})^{-1}P_2\Phi_s \in \tilde{I}^0, \quad s,j = 1,2. \tag{2.16}$$

Remark 2.3. Let $\Phi_{2,1}$ be the solution of the Cauchy problem

$$P_1\Phi_{2,1} \equiv \partial_t\Phi_{2,1} + (\Phi_1^{-1}Q_1\Phi_1 + 2\Phi_1^{-1}\Phi_{1t})\Phi_{2,1} = 0, \quad \Phi_{2,1}(T) = I, \tag{2.17}$$

where I is the identity operator. In this case the operator $\Phi_{2,1}$ can be constructed as a Fourier integral operator modulo smooth operator (see Theorem 3.3 or [3],[4]). More precisely, there exist $\tilde{\Phi}_{2,1}(t) \in I^0$, such that

$$\Phi_{2,1}(t) - \tilde{\Phi}_{2,1}(t) \in B_t(S^{-\infty}). \tag{2.17'}$$

In view of (2.10) and

$$P_2\Phi_2(t) = P_2\left(\Phi_1(t)\int_T^t \Phi_{2,1}(\tau)\,d\tau\right) = P_2\Phi_1(t)\left(\int_T^t (\Phi_{2,1}(\tau)\,d\tau\right) + \Phi_1P_1\Phi_{2,1},$$

or

$$P_2\Phi_2(t) = (P_2\Phi_1(t))\left(\int_T^t \Phi_{2,1}(\tau)\,d\tau,\right), \tag{2.18}$$

the conditions (2.16) can be ignored for $s = 2$. For $s = 1$ the conditions (2.16) reduces to the existence of $\beta(t) \in L_1([0, T])$, such that,

$$\beta^{-1}(t) P \Phi_1 \in \tilde{I}^0. \tag{2.19}$$

Remark 2.4. If the symbol r of the operator

$$R = \Phi_1^{-1}(Q_1 \Phi_1 + 2\Phi_{1t}) \tag{2.20}$$

satisfies the conditions

$$r \in B_t^1(S^1), \tag{2.21}$$

$$\text{principal symbol of } R \text{ is real-valued function for any } t \in (0, T], \quad (x, \xi) \in R^{2n}, \tag{2.22}$$

then the solution of the equation (2.17) satisfies (2.17') (see Theorem 3.3). We define auxiliary operators

$$A_{s,s-1} = \Phi_{s,s-1}^{-1}(t) \partial_t, \quad s = 2, \ldots m, \tag{2.23}$$

and denote by $\sigma = (1, 2, \ldots, m)$ the cyclic permutation from the group of permutations of m numbers. We choose the initial conditions for the equation (2.1) to be

$$\lim_{t \to 0} \{\sigma^j A_{m,m-1} \ldots A_{3,2} A_{2,1} \Phi_1^{-1} u\} = C_{j+1}(y), \quad j = 0, \ldots, m-1, \tag{2.24}$$

where $C_{j+1}(y)$ are arbitrary functions from B and the powers of σ act on indexes of the operators $A_{s,s-1}$ in (2.24).

Theorem 2.2 Let the conditions (2.2), (2.8), (2.12) be satisfied and there exist functions $\mu_k(t, x)$, $\mu(t, x) \in B$ and numbers r_{kj}, $\delta_k \geq 0$, such that

$$\mu_k (\partial_t^k \Phi_1)^{-1} \partial_t^k \Phi_j \in \tilde{I}^{r_{kj}}, \quad k = 0, \ldots, m-1, \quad j = 2, \ldots, m, \tag{2.25}$$

$$\mu_m \Phi_1^{-1} Q_{m-k} (\partial_t^k \Phi_1)^{-1} \mu_k^{-1} \in \tilde{I}^{\delta_k}, \tag{2.26}$$

uniformly in $t \in [0,T]$ (one can put $\delta_k = m$).

Then the initial value problem (2.1), (2.24) for $C_j \in B$, $j = 0, 1, \ldots, m-1$, has a unique solution $u \in B_t^\infty(]0,T])$, and there exists a positive constant c, such that for any solution $u \in B_t^m$ of the problem (2.1), (2.24) the estimates

$$||\mu_k (\partial_t^k \Phi_1)^{-1} \partial_t^k u||_s \leq c \sum_{j=1}^{m} ||C_j||_{s+r_{kj}}, \quad k = 0, \ldots, m-1, \tag{2.27}$$

$$||\mu_m \Phi_1^{-1} \partial_t^m u||_s \leq \sum_{j=1}^{m} ||C_j||_{s+r_{kj}+\delta_k} \tag{2.28}$$

hold.

For the second order equations from Theorem 2.2 follows the following proposition.

Proposition 2.1 Let the conditions (2.2), (2.14) – (2.16) be fulfilled and there exist numbers $\delta, r, b \geq 0$ and functions $\mu_{1,2}(t,x) \in B$, such that

$$\mu_1 \Phi_{1t}^{-1} \Phi_{2t} \in \tilde{I}^r, \quad \mu_2 \Phi_1^{-1} Q_2 \Phi_1 \in \tilde{I}^\delta, \tag{2.29}$$

$$\mu_2 \Phi_1^{-1} Q_1 \Phi_{1t} \mu_1^{-1} \in \tilde{I}^b \tag{2.30}$$

hold. Then the equation

$$P_2 u = (\partial_t^2 + Q_1 \partial_t + Q_2) u = 0 \tag{2.31}$$

with initial data

$$\lim_{t \to 0} \{\Phi_{3-j}^{-1} (\Phi_{1t} \Phi_1^{-1} - \Phi_{2t} \Phi_2^{-1})^{-1} (u_t - \Phi_{jt} \Phi_j^{-1} u)\} = C_j(y), \quad j = 1, 2, \tag{2.32}$$

for $C_j \in B$, has a unique solution $u \in B_t^\infty$, and there exists a positive constant c, such that for every solution u of the problem (2.31), (2.32) the estimates

$$||\mu_k (\partial_t^k \Phi_1)^{-1} \partial_t u||_s \leq c \sum_{j=1}^{2} ||C_j||_{s+r}, \quad k = 0, 1, \quad \mu_0 = 1, \tag{2.33}$$

$$||\mu_2 \Phi_1^{-1} u_{tt}||_s \leq c \sum_{j=1}^{2} ||C_j||_{s+r+\delta} \tag{2.34}$$

hold, where $\mu_0 = 1$.

Consider the equation

$$P_2 u = \left(\partial_t^2 + Q^4(t,x,\partial_x)\right) u(t,x) = 0, \quad t \in]0,T[, \quad x \in R^n, \tag{2.35}$$

where Q is invertible in $H^{-\infty} = \cup H^s$ elliptic pseudodifferential operator of order 1/2 infinitely differentiable by parameter $t \in]0,T]$. Let $\Phi_j(t,T)$, $j = 1,2$ be the resolving Fourier integral operators of the first order pseudodifferential equation

$$\partial_t w_j + \Lambda_j(t,x,\partial_x) w_j(t,x) = 0, \quad t \in]0,T[, \quad x \in R^n, \tag{2.36}$$

where the operators Λ_1, Λ_2 (with symbols λ_1, λ_2) are defined to be

$$\Lambda_1 = iQ^2 - Q^{-1}Q_t, \quad \Lambda_2 = -iQ^2 - Q^{-1}Q_t, \tag{2.37}$$

that is, the solutions $w_j(t,x)$ of the equation (2.36) is representable in the form

$$w_j(t,x) = \Phi_j(t,T) w_j(T,x), \quad j = 1,2. \tag{2.38}$$

Proposition 2.2 Let there exist functions $\rho_1(t), \rho_2(t) \geq 0$; $\mu(t), \rho_1(t), \rho_2(t) \in L_1[\varepsilon,T]$, $\varepsilon > 0$, such that

$$\sigma(\mu(t)Q^2) \in B_t^\infty(S^1), \quad \sigma(\mu Q^{-1}Q_t) \in B_t^\infty(S^0), \tag{2.39}$$

$$P_2 \text{ is } t\text{-hyperbolic operator}, \tag{2.40}$$

$$\rho_1^{-1}(t) Q^{-3}[Q_t, Q]Q, \quad \rho_2^{-1}(t) Q^{-2}(Q^{-1})_{tt} Q \in \Psi_\rho^0, \tag{2.41}$$

uniformly in $t \in [0,T]$. Then for every choice of $C_j(y) \in B$, $j = 1,2$ there exists a solution $u \in B_t^\infty$ of the equation (2.35) that can be written in the form (2.13), (2.13') with $m = 2$, where Φ_1, Φ_2 are the resolving operators of the equation (2.36).

Remark 2.5. The condition (2.40) for the equation (2.35) means, that the principal symbol of Q^2 is real-valued on $[0, T] \times R^{2n}_{x,\xi}$.

Denote by $|| \cdot ||_{\Psi^m_\rho}$ the norm of pseudodifferential operator from the class Ψ^m_ρ: $|| \cdot ||_{\Psi^m_\rho} = C|p|^{(m)}_l$, see Theorem 3.2 in §3.

Proposition 2.3 Under the conditions of Proposition 2.2, if

$$||Q^{-3}Q_t||_{\Psi^0_\rho} < 1 \quad \text{uniformly by } t \in [0, T], \tag{2.42}$$

then the equation (2.35) with $C_1, C_2 \in H^p$ and initial data

$$\lim_{t \to 0} \Phi_2(T, t)(Q^{-2}\partial_t + i - Q^{-3}Q_t)u = C_1(x),$$

$$\lim_{t \to 0} \Phi_1(T, t)(Q^{-2}\partial_t - i - Q^{-3}Q_t)u = C_2(x), \tag{2.43}$$

has unique solution $u \in B^\infty_t$. A positive constant c exists which does not depend on u, such that

$$||\mu_0 \Phi_1^{-1} u||_s, \quad ||\mu_1 \Phi_{1t}^{-1} u_t||_s \le c \sum_{j=1}^{2} ||C_j||_{s+2}, \quad ||\mu^2 \Phi^{-1} u_{tt}||_s \le c \sum_{j=1}^{2} ||C_j||_s. \tag{2.44}$$

Introduce the class

$$M = \{p(t) \in C^\infty(]0, T]), \quad p(t) > 0, \quad p_t p^{-5/4} \in L_2([0, T]), \quad |p_t p^{-3/2}| \le \varepsilon\}, \tag{2.45}$$

where ε is sufficiently small positive number.

Proposition 2.4 Let there exist a function $p(t) \in M$, such that the conditions (2.39), (2.40) and

$$p^{1/4}(t)Q^{-1} \in \Psi^{-1/2}, \quad (p/p_t)^2[Q, Q_t] \in \Psi^1, \tag{2.46}$$

$$p^{-1/4}(p/p_t)^j(\partial_t^j Q) \in \Psi^{1/2}, \quad j = 0, 1, 2 \tag{2.47}$$

are satisfied, where $[\cdot, \cdot]$ is the commutator. Then the problem (2.35), (2.43) for $C_1 \in H^p$, $C_2 \in H^{p-1}$ has unique solution $u \in B_t^\infty$, and there exists a positive constant c, such that for every solution u of this problem the estimates

$$\tau(t)p^{1/4-k/2}(t)||\partial_t^k u||_s \leq \sum_{j=1}^{2} ||C_j||_s, \quad k = 0, 1, 2 \tag{2.48}$$

hold. Here

$$\tau(t) = \exp\left(\int_T^t (p(s))^{1/2} ds\right).$$

Remark 2.6. If we replace the condition (2.40) by

$$Q^2 - (Q^2)^* \in \Psi^0, \tag{2.49}$$

then the estimates (2.48) simplifies:

$$p^{1/4-k/2}(t)||\partial_t^k u||_s \leq c\sum_{j=1}^{2} ||C_j||_{s+k}, \quad k = 0, 1, 2. \tag{2.50}$$

Proposition 2.4 has been proved in the [8] by a different method. We can obtain different asymptotic solutions of the equation

$$u_{tt} + Qu = 0, \quad t \in]0, T[, \quad x \in R^n, \quad Q \in \Psi^2, \tag{2.51}$$

if we represent the solutions of (2.51) in the form $u = \Phi(t)v(t, x)$, where Φ is the resolving Fourier integral operator of the first order pseudodifferential equation

$$\{\, \Phi_t = (A_1 + \cdots + A_r)\Phi, \quad t \in]O, T[, \Phi\big|_{t=T} = I.$$

Above I is the identity operator and A_i are the pseudodifferential operators, that can be found from recurrent relations

$$A_{1t} + Q = 0,$$

$$A_{jt} + A_{j-1}^2 + A_j(A_1 + \cdots + A_{j-1}) + (A_1 + \cdots + A_{j-1})A_j = 0, \quad j = 2, .., r. \quad (2.52)$$

From (2.52) it is easy to deduce, that

$$\Phi_{tt} + Q\Phi = A_r^2. \quad (2.53)$$

Let $\Phi_{2,1}$ be the solution of the first order operator equation

$$(\partial_t + 2\Phi^{-1}\Phi_t)\Phi_{2,1} = 0, \quad \Phi_{2,1}\big|_{t=0} = I,$$

where

$$\Phi_2(t) = \Phi(t) \int_T^t \Phi_{2,1}(\tau)\, d\tau.$$

Proposition 2.5 Let there exist a non-negative function $\beta(t)$ from the class $L_1([0, T])$, such that

$$\Phi, \Phi^{-1}, \Phi_{2,1}, \Phi_{2,1}^{-1} \in I^0, \qquad \beta^{-1}(t)A_r^2 \in \Psi_\rho^0,$$

and principal symbol of Q is non-negative function on $[0, T] \times R^{2n}$. Then for $C_1(x), C_2(x) \in H^\infty$ a solution of the equation (2.53) exists, representable in the form

$$\partial_t^k u = [\Phi^{(k)}(t) + \varepsilon_1(t, x)]C_1(x) + [\Phi_2^{(k)}(t) + \varepsilon_2(t, x)]C_2(x),$$

where

$$\lim_{t \to 0} \varepsilon_k(t, .) = 0, \quad k = 0, 1, \quad \text{uniformly by } x \in R^n.$$

§3. PROOF OF THEOREM 2.1

Assume that

$$\lambda_1(t,x,\xi) = \tilde{\lambda}_1(t,x,\xi) + \lambda(t,x,\xi), \tag{3.1}$$

$$\tilde{\lambda}_1(t,x,\xi) \in B_t^1(S^1), \quad \lambda \in B_t^1(S^0), \quad \tilde{\lambda}_1 \text{ is real-valued.} \tag{3.2}$$

We are looking for the phase function of the operator (2.5), $I(t,s) = I(t,s;x,\xi) \in B_t^2(S^1)$ on $0 \le s \le t \le T_0$ for some T_0 $(0 < T_0 \le T)$ as a solution of the Cauchy problem

$$\partial_t \varphi - \lambda_1(t,x,\nabla_x \varphi) = 0, \quad \text{on} \quad 0 \le s \le t \le T_0, \tag{3.3}$$

$$\varphi(s,s;x,\xi) = x\xi. \tag{3.3$'$}$$

Equation (3.3) is called *the eikonal equation.* For $\{q,p\} = \{(q_1,\ldots,q_n),(p_1,\ldots,p_n)\}$ we consider the system of Hamilton equations

$$\frac{\partial q}{\partial t} = -\nabla_p \lambda_1(t,q,p), \quad \frac{\partial p}{\partial t} = \nabla_q \lambda_1(t,q,p), \quad \{q,p\}_{t=s} = \{y,\eta\}. \tag{3.3$''$}$$

Theorem 3.1 ([4], Theorem 3.1) Let $y = y(t,s;x,\xi)$ $(0 \le s \le t \le T_2)$ be the inverse mapping of $x = q(t,s;y,\xi) : y \longmapsto x$, $R^n \longmapsto R^n_x$ with (t,s,ξ) a parameter, and define

$$u(t,s;y,\eta) - y\eta = \int_s^t [\lambda_1 - p\nabla_\xi \lambda_1](\tau, q(\tau,s;y,\eta), p(\tau,s;y,\eta)\, d\tau. \tag{3.4}$$

Then the solution of Cauchy problem (3.3), (3.3') can be written as $I(t,s;x,\xi) = u(t,s;y(t,s;x,\xi),\xi)$. Setting $J(t,s) = J(t,s;x,\xi) = \varphi(t,s;x,\xi) - x\xi$, we have

$$\{J(t,s)/(t-s), \partial_t J(t,s), \partial_s J(t,s)\}_{0 \le t,s \le T_0} \quad \text{is bounded in } S^1. \tag{3.5}$$

Corollary 3.1. (1) For any integer $\nu \ge 0$ there exist c_ν and T_ν $(0 < T_\nu \le T_0)$ such that

$$I(t,s;x,\xi) \in \mathcal{P}(c_\nu|t-s|,\nu), \quad 0 \le t,s \le T_\nu. \tag{3.5$'$}$$

(2) If in Theorem 3.1, for $1 \leq m < \infty$

$$\lambda(t,x,\xi) \in \mathcal{B}_t^m(S^1) \quad \text{on} \quad [0,T],$$

then

$$\{J(t,s)/(t-s), \partial_t^j \partial_s^k J(t,s); j+k \leq m+1\}_{0 \leq t,s \leq T_0}$$

is bounded in S^1.

Theorem 3.2 ([4], Theorem 2.3) Let $P \in I_\rho^m$. Then for any real s the operator P defines continuous mappings from H^{s+m} into H^s. There exist a constant $C = C > 0$ and an integer $l = l(s,m) \geq 0$ such that

$$||Pu||_s \leq C|p|_l^{(m)}||u||_{s+m}, \quad \text{for} \quad u \in H^{s+m}. \tag{3.6}$$

Theorem 3.3 ([4], Theorem 3.2) Consider the operator $L = \partial_t - \Lambda(t,x,D_x)$. Assume that $\lambda = \sigma(\Lambda)$ satisfies the conditions (3.1), (3.2). Then there exists a linear integral operator $E(t,s) \in \tilde{I}^0$, such that

$$E(s,s) = I, \quad L_t E(t,s) = 0 \quad \text{on} \quad 0 \leq s \leq t \leq T_0.$$

The operator $E(t,s)$ is Fourier integral operator modulo smooth operators: there exist an operator $\widetilde{E}$ with a symbol $e(t,s;x,\xi) \in B_t^1(S^0)$ $(0 \leq s \leq t \leq T_0$ with T_0 as in Corollary 3.1), such that

$$\tilde{E}_\varphi(t,s) \in I^0, \quad \tilde{E}(t,s) - E(t,s) \in B_t(S^{-\infty}).$$

Denote auxiliary functions $\{u^j(t,y)\}_{j=1}^m$ by the expressions

$$\partial_t^{k-1} u(t,x) = \sum_{j=1}^m [\partial_t^{k-1}\Phi_j(t)] u^j(t,y), \quad k = 1, \dots, m. \tag{3.7}$$

Resolving the relations (3.7) by u^j, we have

$$u^j(t,y) = \sigma^j A_{m,m-1} \cdots A_{3,2} A_{2,1} \Phi_1^{-1} u, \quad j = 1, \dots, m, \tag{3.8}$$

where the linear integral operators $A_{s,s-1}$ are defined from (2.23), and σ is the cyclic permutation from the group of permutations of m numbers $(1,\ldots,m)$. Let us prove (3.8) in the particular case $m=3$. We eliminate $u^1(t,y)$ from the system

$$u = \Phi_1 u^1 + \Phi_2 u^2 + \Phi_3 u^3,$$

$$u_t = \Phi_{1t} u^1 + \Phi_{2t} u^2 + \Phi_{3t} u^3,$$

$$u_{tt} = \Phi_{1tt} u^1 + \Phi_{2tt} u^2 + \Phi_{3tt} u^3 \tag{3.7$'$}$$

applying $(\Phi_1^{-1})_t$ to the left side of the first relation and Φ_1^{-1} to the left side of the second relation. After adding these expressions we get

$$(\Phi_1^{-1}\Phi_2)_t u^2 + (\Phi_1^{-1}\Phi_3)_t u^3 = (\Phi_1^{-1}u)_t.$$

Applying $(\Phi_1^{-1})_{tt}$ to the left side of the first relation in (3.7'), $(2\Phi_1^{-1})_t$ to the second, and Φ_1^{-1} to the third relation in (3.7') and adding we get

$$(\Phi_1^{-1}\Phi_2)_{tt} u^2 + (\Phi_1^{-1}\Phi_3)_{tt} u^3 = (\Phi_1^{-1}u)_{tt}.$$

That is, we get

$$\Phi_{2,1t} u^2 + \Phi_{3,1t} u^3 = (\Phi_1^{-1}u)_{tt}, \quad \Phi_{2,1} u^2 + \Phi_{3,1} u^3 = (\Phi_1^{-1}u)_t. \tag{3.9}$$

Now we eliminate u^2 from (3.9) applying Φ_{21}^{-1} to the first relation in (3.9), $(\Phi_{21}^{-1})_t$ to the second relation and adding, we get

$$(\Phi_{2,1}^{-1}\Phi_{3,1})_t u^3 = (\Phi_{2,1}^{-1}(\Phi_1^{-1}u_t)_t),$$

or

$$u^3 = \Phi_{3,2}^{-1}\partial_t(\Phi_{2,1}^{-1}\partial_t(\Phi_1^{-1}u)) = A_{3,2}A_{2,1}\Phi_1^{-1}u, \tag{3.10}$$

where

$$\Phi_2(t) = \Phi_1(t)\int_T^t \Phi_{2,1}(y)\,dy, \quad \Phi_3(t) = \Phi_1(t)\int_T^t \Phi_{3,1}(y)\,dy,$$

$$\Phi_{3,1} = \Phi_{2,1}(t)\int_T^t \Phi_{3,2}(y)\,dy, \quad A_{3,2} = \Phi_{3,2}^{-1}\partial_t, \quad A_{2,1} = \Phi_{2,1}^{-1}\partial_t. \qquad (3.11)$$

By index permutation we have

$$u^1 = (1,2,3)u^3 = A_{1,3}A_{3,2}\Phi_2^{-1}u, \quad u^2 = (1,2,3)^2u^3 = A_{2,1}A_{1,3}\Phi_3^{-1}u. \quad (3.12)$$

The relations (3.8) in general case are proved similarly. Differentiating (3.7) in t, in view of (2.1), we get

$$\sum_{j=1}^m (\partial_t\Phi_j^k)\partial_t u^j = -\delta_{km}\sum_{j=1}^m (P_j)u^j, \quad k = 1,\dots,m, \qquad (3.13)$$

where δ_{km} is Kronecker symbol. The relations (3.13), in view of (2.9) – (2.9"), become

$$\partial_t u^s + \sum_{j=1}^{m-s} Z_{j+s,s}(t)\partial_t u^{j+s} = 0, \quad s = 1,\dots,m-1,$$

$$\partial_t u^m = -(\Phi_{2,1}\Phi_{3,2}\dots\Phi_{m,m-1})^{-1}\sum_{j=1}^m (P\Phi_j)u^j. \qquad (3.14)$$

Resolving the relations (3.14) relative to $\partial_t u^s$, we get

$$\partial_t u^s = \sum_{j=1}^m K_j^s u, \quad s = 1,\dots,m, \qquad (3.15)$$

where the linear integral operators K_j^s are defined by (2.11). We show how to obtain (2.11) in particular case $m = 3$. By differentiation (3.7') in t we have

$$\Phi_1 u_t^1 + \Phi_2 u_t^2 + \Phi_3 u_t^3 = 0, \quad \Phi_{1t}u_t^1 + \Phi_{2t}u_t^2 + \Phi_{3t}u^3 = 0,$$

$$\Phi_{1tt}u_t^1 + \Phi_{2tt}u_t^2 + \Phi_{3tt}u_t^3 = -(P_3\Phi_j)u^j,$$

where

$$P_3\Phi_j = (\partial_t^3 + Q_1\partial_t^2 + Q_2\partial_t + Q_3)\Phi_j.$$

In view of

$$\Phi_2 = \Phi_1 Z_{2,1}, \quad \Phi_3 = \Phi_1 Z_{3,1}, \quad Z_{3,1} = \int_T^t \Phi_{3,1}(s)\, ds, \quad \Phi_{3,1} = \Phi_{2,1} Z_{3,2},$$

(see (2.9) – (2.9")) we obtain

$$u_t^1 + Z_{2,1}u_t^2 + Z_{3,1}u_t^3 = 0, \quad u_t^2 + Z_{3,2}u_t^3 = 0, \quad \Phi_1\Phi_{2,1}\Phi_{3,2}u_t^3 = -P_3\Phi_j u^j,$$

that is (3.15), where

$$K_j^3 = -(\Phi_1\Phi_{2,1}\Phi_{3,2})^{-1}P_3\Phi_j,$$

$$K_j^1 = (Z_{2,1}Z_{3,2} - Z_{3,1})K_j^3, \quad K_j^2 = -Z_{3,2}K_j^3. \tag{2.11$'$}$$

The proof of (2.11) in general case can be proved in similar way. From (3.15), by integration in $t \in [0, T]$ we obtain a system of integrodifferential equations

$$u^s(t, y) = C_s(y) + \int_T^t \sum_{j=1}^m K_j^s(\tau)u^j(\tau, y)d\tau, \quad s = 1, \ldots, m. \tag{3.16}$$

Assuming $\beta_s^{-1}(t)K_j^s \in \tilde{I}^0$, see (2.12), we obtain the estimates

$$||K_j^s v||_p \leq c\beta_s(t)||v||_p, \quad s, j = 1, \ldots. \tag{3.17}$$

From (3.16), (3.17) we get

$$||u^s(t)||_p \leq ||C_s||_p + c\int_0^t \beta_s(\tau) \sum_{j=1}^m ||u^j(\tau)||_p\, d\tau. \tag{3.18}$$

Summation over s yields

$$\sum_{s=1}^m ||u^s(t)||_p \leq \sum_{s=1}^m ||C_s||_p + c\int_0^t \beta(\tau) \sum_{j=1}^m ||u^j(\tau)||_p\, d\tau, \tag{3.19}$$

where

$$\beta(t) = \sum_{s=1}^m \beta_s(t).$$

Applying Gronwall inequality to (3.19), we find

$$\sum_{s=1}^{m} ||u^s(t)||_p \leq \sum_{s=1}^{m} ||C_s|| \exp\left(c\int_0^t \beta(\tau)\, d\tau\right). \tag{3.20}$$

Again applying (3.16), in view of (3.20), we get

$$||u^s - C_s||_p \leq c\int_0^t \left(\beta_s(\tau)\exp\int_0^\tau c\beta(z)dz\right) d\tau \leq c\int_0^t \left(\beta(\tau)\exp\int_0^\tau c\beta(z)dz\right) d\tau,$$

or, by integration

$$||u^s - C_s||_p \leq c_1 \left[-1 + \exp\left(c\int_0^t \beta(y)\, dy\right)\right], \quad s = 1, \dots m. \tag{3.21}$$

Denoting

$$\varepsilon_j(t,y) = u^j(t,y) - C_j(y), \quad j = 1, \dots, m, \tag{3.22}$$

from (3.21) we get the relations (2.13), (2.13'). The existence of a solution of the equation (2.1) can be proved applying the iterations to (3.16). The proof of Theorem 2.1 is complete.

§4. PROOFS OF THEOREM 2.2 AND PROPOSITIONS

Proof of Theorem 2.2 Under the assumptions of Theorem 2.2, from Theorem 2.1 we have the representations (2.13), (2.13') and the relations

$$\mu_k(\partial_t^k \Phi_1)^{-1}\partial_t^k u = \mu_k(C_1+\varepsilon_1)+\sum_{j=2}^{m-1} \mu_k(\partial_t^k \Phi_1)^{-1}\partial_t^k \Phi_j(C_j+\varepsilon_j), \quad k = 0, 1, \dots, m-1. \tag{4.1}$$

Using the condition (2.25), we get the estimates (2.27). From the equation (2.1) we have

$$\partial_t^m u = -\sum_{k=0}^{m-1} Q_{m-k}\partial_t^k u,$$

and

$$\mu_m \Phi_1^{-1} \partial_t^m u = - \sum_{k=0}^{m-1} \mu_m \Phi_1^{-1} Q_{m-k} (\partial_t^k \Phi_1) \mu_k^{-1} \mu_k (\partial_t^k \Phi_1)^{-1} \partial_t^k u. \tag{4.2}$$

In view of (2.25), (2.26), we get the estimates (2.28):

$$||\mu_m \Phi^{-1} \partial_t^m u||_s \le \sum_{k=0}^{m-1} ||\mu_k (\partial_t^k \Phi_1)^{-1} \partial_t^k u||_{s+\delta_k} \le \sum_{k=0}^{m-1} \sum_{j=1}^{m} ||C^j||_{s+r_{kj}+\delta_k}.$$

Theorem 2.2 is proved.

Proposition 2.1 is a direct corollary of Theorem 2.1. Another proof can be found in [8].

Proof of Proposition 2.2 From (2.36) – (2.38) follows that linear integral operators Φ_1, Φ_2 are solutions of the Cauchy problems

$$\partial_t \Phi_j + \Lambda_j \Phi_j = 0, \quad t \in]0, T[, \quad x \in R^n,$$

$$\Phi_j \big|_{t=T} = I, \quad j = 1, 2. \tag{4.3}$$

From (2.36), (2.37) we obtain

$$\Phi_{2t} \Phi_2^{-1} - \Phi_{1t} \Phi_1^{-1} = \Lambda_1 - \Lambda_2 = 2iQ^2,$$

$$\Lambda_j^2 - \Lambda_{jt} = 2(Q^{-1} Q_t)^2 - Q^{-1} Q_{tt} \pm iQ^{-1} [Q_t, Q] Q,$$

$$P\Phi_j = \Phi_{jtt} + Q^4 \Phi_j = (\Lambda_j^2 - \Lambda_{jt} + Q^4) \Phi_j = \{2(Q^{-1} Q_t)^2 -$$

$$-Q^{-1} Q_{tt} \pm iQ^{-1} [Q, Q_t] Q\} \Phi_j = \{(Q^{-1})_{tt} \pm iQ^{-1} [Q, Q_t]\} Q \Phi_j, \quad j = 1, 2. \tag{4.4}$$

These relations transform the conditions (2.16) of Remark 2.2 to

$$\beta_s^{-1}(t) \Phi_j^{-1} Q^{-2} \{(Q^{-1})_{tt} \pm iQ^{-1} [Q, Q_t] Q \Phi_s \in \tilde{I}^0, \quad s = 1, 2.$$

They are satisfied in view of the conditions (2.41) of Proposition 2.2. From the conditions (2.39), (2.40) of Proposition 2.2 follow the conditions (2.14), (2.15) of

Remark 2.2 ($Q_1 = 0, Q_2 = Q^4$). The operator $\Phi_1 \in I^0$ modulo smooth operator (see Theorem 3.3), since Φ_1 satisfies the first order pseudodifferential equation $\Phi_{1t} = -\Lambda_1\Phi_1$, where the symbol λ_1 of Λ_1 satisfies the conditions (3.1), (3.2). Similarly, $\Phi_{2,1} \in I^0$, modulo smooth operator, because Φ_{21} satisfies the first order pseudodifferential equation (2.17). So the conditions of Proposition 2.2 imply all conditions of Remark 2.2. Proposition 2.2 is a consequence of Theorem 2.1 and Remark 2.2.

Proof of Proposition 2.3 Let us prove that the conditions of Propositions 2.3 imply the conditions (2.29), (2.30) of Proposition 2.1. Because $\Phi_{jt} = -\Lambda_j\Phi_j$, the condition $(2.29)_1$ become

$$\mu_1\Phi_{1t}^{-1}\Phi_{2t} = \mu_1\Phi_1^{-1}\Lambda_1^{-1}\Lambda_2\Phi_2 \in \tilde{I}^r, \quad \text{with } r = 0.$$

This relation follows from the condition $\Lambda_1^{-1}\Lambda_2 \in \Psi_\rho^0$, which is the consequence of (2.42) and of

$$(iQ^2 + Q^{-1}Q_t)^{-1}(-\mathrm{i}Q^2 + Q^{-1}Q_t) = (i + Q^{-3}Q_1)^{-1}(-i + Q^{-3}Q_t) \in \Psi_\rho^0.$$

The condition $(2.29)_2$ with $\delta_0 = 2$, $\mu = \mu^2$ follows from (2.39): $\mu Q^2 \in \Psi_\rho^1$ implies $\mu^2Q^4 = (\mu Q^2)^2 \in \Psi_\rho^2$, and

$$\mu_2\Phi_1^{-1}Q_2\Phi_1 = \mu^2\Phi_1Q^4\Phi_1 \in \Psi_\rho^2 \subset I_\rho^2.$$

The condition (2.30) is satisfied, because of $Q_1 = 0$. Thus the conditions of Proposition 2.1 are satisfied with $\mu_2 = \mu^2(t), r = 0, \delta_0 = 2, \mu_1 = 1$. In view of

$$\Phi_{1t}\Phi_1^{-1} - \Phi_{2t}\Phi_2^{-1} = \Lambda_2 - \Lambda_1 = -2iQ^2, \quad u_t - \Phi_{jt}\Phi_j^{-1}u_t = u_t + \Lambda_j u, \tag{4.5}$$

initial data (2.32) transform to (2.43). From the estimates (2.33) of Proposition 2.1 we get (2.44). So Proposition 2.3 follows from Proposition 2.1.

Lemma 4.1 Let the conditions (2.39), (2.49) are satisfied. Then the solutions w of the equations

$$\partial_t w + \Lambda_j w = 0, \quad t \in]0, T[\tag{4.6}$$

satisfy the estimates

$$||Q(t)w(t)||_s \exp\left(c\int_T^t \mu^{-1}(y)\, dy\right) \leq ||Q(T)w(T)||_s \leq ||Q(t)w(t)||_s \exp\left(c\int_T^t \mu^{-1}(y)\, dy\right). \tag{4.7}$$

Proof The conditions (2.39), (2.40) imply that

$$\mu(t)[Q^2 - (Q^2)^*] \in \Psi^0_\rho.$$

By substitution $v = Qw$, $\tau = \tau(t) = \int_T^t \mu^{-1}(y)\, dy\}$ in (4.6), we obtain

$$v_t \pm iQ^2 v = 0, \qquad \text{or } v_\tau \pm i\mu Q^2 = 0,$$

and

$$\bar{v}v_\tau + i\bar{v}\mu Q^2 v = 0, \quad v\bar{v}_\tau \pm iv\bar{Q^2}v = 0.$$

We get

$$\partial_\tau ||v||^2 + i(v, \mu Q^2 v) \pm i(\mu Q^2 v, v) = 0,$$

or

$$\partial_\tau ||v||^2 = \pm i(v\mu(Q^2 - Q^{2*})v).$$

The assumption (2.49) implies $|\ \partial_\tau ||v||^2\ | \leq c||v||^2$, or, letting $v \to E_s v = (1+ |\ D_x\ |^2)^{s/2} v$, we have

$$|\partial_\tau ||v||_s| \leq c||v||_s \quad \text{ or } -cd\tau \leq ||v||_s^{-1} d||v||_s \leq cd\tau.$$

By integration over $\tau \in [\tau, 0], \tau \leq 0$ we get

$$-c\int_\tau^0 d\tau \leq \ln\frac{||v(0)||_s}{||v(\tau)||_s} \leq c\int_\tau^0 d\tau,$$

$$||v(\tau)||_s \exp(c\tau) \leq ||v(0)||_s \leq ||v(\tau)||_s \exp(-c\tau),$$

$$||v(t)||_s \exp\{c\int_T^t \mu^{-1}(y)dy \leq ||v(T)||_s \leq ||v(t)||_s \exp\{c\int_t^T \mu^{-1}(y)dy.$$

Because $v = Qw$, we obtain the estimate (4.7).

Proof of Proposition 2.4 First we prove the estimates (2.48). From Lemma 4.1 we obtain

$$||\Phi_1^{-1}u(t)||_s = ||u(T)||_s = ||Q^{-1}(T)Q(T)u(T)||_s \geq ||Q(T)u(T)||_{s-1/2} \geq$$

$$\geq ||Q(T)u(t)||_{s-1/2} \exp\{c\int_T^t \mu^{-1}(y)\,dy\} \geq p^{1/4}(t)\exp\{c\int_T^t \mu^{-1}(y)\}\,dy\}||u(t)||_s,$$

therefore, in view of (2.44) we get (2.48) for $k = 0$:

$$p^{1/4}(t)\{\exp\{c\int_T^t \mu^{-1}(y)\}\,dy\}\}||u(t)||_s \leq ||\Phi_1^{-1}u(t)||_s \leq c\sum_{j=1}^{2}||C^j||_s.$$

Furthermore

$$\mu p^{1/4}(t)\{\exp\{c\int_T^t \mu^{-1}(y)\,dy\}\}||u(t)||_s \leq \mu||\Phi_1^{-1}u_t(t)||_s \leq c\sum_{j=1}^{2}||C_j||_s.$$

Because $\mu = p^{-1/2}$, we obtain (2.48) for $k = 1$. The proof of (2.48) is completed by the observation

$$\mu^2 p^{1/4}(t)\{\exp\{c\int_T^t \mu^{-1}(y)\}\,dy\}\}||u_{tt}||_s \leq \mu^2||\Phi_1^{-1}u_{tt}(t)||_s \leq c\sum_{j=1}^{2}||C_j||_s.$$

To prove Proposition 2.4 we show, that the conditions of Proposition 2.3 follow from the assumptions of Proposition 2.4. Putting $j = 0$ in (2.47) we get $p^{-1/4}Q \in \Psi^{1/2}$; this implies $\mu Q^2 = (p^{-1/4}(t)Q)^2 \in \Psi^1$, that is, $(2.39)_1$ with $\mu = p^{-1/2}(t)$. Putting $j = 1$ in (2.47), we get from (2.46) $p^{1/4}Q^{-1} \in \Psi^{-1/2}$, $p_t^{-1}p^{3/4}Q_t \in \Psi^{1/2}$; this implies

$$p^{-1/2}Q^{-1}Q_t = p^{-3/2}p_t(p^{1/4}Q^{-1})(p_t^{-1}p^{3/4}Q_t) \in \Psi^0$$

(in view of the inequality (2.45) $\mid p^{-3/2}p_t \mid < \varepsilon$), that is (2.39). From (2.46), (2.47) with $\rho = p^{-5/2}p_t^2 \in L_1([0,T])$, (see (2.45)), we obtain the condition (2.41) as follows

$$\rho_1 Q^{-3}[Q_t, Q]Q = (p^{1/4}Q^{-1})^3 p_t^{-2} p^2 [Q_t, Q](p^{-1/4}Q) \in \Psi^0,$$

$$\rho_2(t)Q^{-2}(Q^{-1})_{tt}Q = \rho_2^{-1}(t)Q^{-2}[2(Q^{-1}Q_t)^2 - Q^{-1}Q_{tt}] =$$

$$= (p^{1/4}Q^{-1})^2[2(p^{1/4}Q^{-1}p^{3/4}p_t^{-1}Q_t) - (p^{1/4}Q^{-1})p^{7/4}p_t^{-2}Q_{tt}] \in \Psi^0,$$

since

$$(Q^{-1})_{tt}Q = 2(Q^{-1}Q_t) - Q^{-1}Q_{tt}.$$

In view of (2.45), the condition (2.42) follows from the inclusion

$$Q^{-3}Q_t = (p^{1/4}Q^{-1})^3(p^{3/4}p_t^{-1}Q_t)p_t p^{-3/2} \in \Psi^{-1}.$$

Proposition 2.4 is proved.

To prove the Remark 2.6, we note, that in view of (2.48), (2.39), we can put in (4.7) $\mu = 1$. Therefore the weight function $\exp\{c\int_T^t \mu^{-1}(y)\}\ dy$ tends to a constant, as t tends to 0 and (2.50) follows from the estimates (4.7) of Lemma 4.1.

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16 February 1996

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