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LEVINSON THEOREM FOR 2 X 2 SYSTEM AND APPLICATIONS TO THE ASYMPTOTIC STABILITY AND SCHRODINGER EQUATION

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Abstract

We prove new asymptotical stability and instability theorems for non autonomous 2×2 system of first-order differential equations by using a new version of the classical Levinson asymptotic theorem for 2×2 systems. The proof of this version is based on the construction of approximate fundamental solution of the original system in the special form with unknown phase function and the error estimates formulated in the terms of generalized characteristic functional. In the case of constant matrix A generalized characteristic functional turns to the usual characteristic polynomial and by choosing phase functions as eigenvalues of the matrix A the error could be eliminated. As another application we derive a transition probability formula for the two level atom in the external electromagnetic field described by Schrodinger system.

Key words: Asymptotic stability, asymptotic solutions, characteristic function, fundamental matrix, integral representation, stability estimates, first order system of differential equations, Schrodinger equation, two level atom, transition probability

1. Main Results

Consider the system of linear ordinary differential equations

$$u'(t) = A(t)u(t), \quad t > T, \tag{1.1}$$

where $u(t)$ is a 2-vector function, and

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$$

is a 2×2 matrix-function differentiable by $t \in (T,\infty)$.

The rest state $u(t) = 0$ of (1.1) is called stable if for any $\varepsilon > 0$ there exists $\delta(T,\varepsilon) > 0$ such that if $|u(T)| < \delta(T,\varepsilon)$ then $|u(t)| < \varepsilon$ for all $t \geq T$. The rest state $u(t) = 0$ of (1.1) is called asymptotically stable if it is stable, and attractive:

$$\lim_{t\to\infty} u(t) = 0 \tag{1.2}$$

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for every solution of (1.1).

The usual method of investigation of asymptotic stability of differential equations is the Lyapunov's method that uses energy functions and Lyapunov stability theorems [2, 3, 5, 13, 6, 8, 12, 14, 16, 17].

Here we continue the development of another approach started in [9, 10, 11]. This approach based on the usage of different asymptotic solutions [15] investigated in [7] (instead of construction of energy functions in Lyapunov's method), and the error estimates [4]. To prove stability inequalities for system (1.1) we use a new version of the Levinson theorem (see Theorem 2.1 below) for 2×2 systems about asymptotic solutions with explicit estimate of the error term, which may be used also for finding actual asymptotic solutions (see Remark 2.2). The classical Levinson theorem uses a decomposition of the right side matrix function $A = B+R$, where the leading matrix B is diagonal and the perturbation matrix R is integrable. In our version we prove the error estimate using the decomposition with the leading matrix B such that corresponding system is explicitly solvable. To prove the error estimates we use a construction of approximate fundamental matrix of system (1.1) in the special form with an unknown phase function $\varphi(t)$, which may be chosen by using known asymptotic solutions. In the paper we illustrate on examples some choices of the function $\varphi(t)$. For instance one of the choices of $\varphi(t)$ is based on the Green-Liouville asymptotic solutions (see (1.30))

Examples show that asymptotic solutions approach works better than Lyapunov's method for the systems with complex valued coefficients (see Example 1.3).

There is a bridge connecting asymptotic solutions approach with Lyapunov's method: when the asymptotic fundamental matrix solution Ψ of (1.1) is chosen the appropriate energy function of Lyapunov may be constructed by the formula

$$E(t,u(t)) = \|\Psi^{-1}(t)u(t)\|^2.$$

Indeed $E(t)\geq 0$, and if Ψ is the exact fundamental matrix function of (1.1) then conservation law $E'(t) = 0$ is true.

Furthermore we deduce the transition probability formula for the Schrodinger system that describes the interaction of two-level atom with electromagnetic field, and we give the comparison of two approximate solutions.

Denote by $L^1(T,\infty)$ the class of Lebesgue integrable in (T,∞) functions and by $C^1(T,\infty)$ the class of differentiable functions on (T,∞).

Denote

$$TrA(t) = a_{11}(t)+a_{22}(t), \quad |A(t)| = det(A(t)).$$

Asymptotic behavior of solutions of autonomous systems is described by eigenvalues of corresponding matrix A. The key step of finding behavior of solutions of non autonomous system (1.1) is to find the phase functions θ_j that are minimizing (or eliminating) the generalized characteristic functional

$$Char(\theta) = -\theta^2-\theta'+\theta\left(Tr(A)+\frac{a'_{12}}{a_{12}}\right)-|A|-\frac{W[a_{11},a_{12}]}{a_{12}}, \tag{1.3}$$

where $W[\cdot,\cdot]$ is a Wronskian:

$$W[a,b] = a(t)b'(t)-a'(t)b(t).$$

Here and further in the text we often suppressed dependence on t for simplicity.

Note that in the case of constant matrix A in (1.1) the characteristic functional (1.3) turns to the usual characteristic polynomial:

$$Char(\theta) = -\theta^2 + \theta TrA - |A|,$$

so we can eliminate the characteristic functional by choosing phase functions as eigenvalues of the matrix A.

Using Liouville's formula that gives the connection between the functions θ_j we can start from a single unknown phase function $\xi(t) \in C^2[T,\infty)$, and the matrix function A(t) to construct the phase functions θ_j :

$$\theta_{1,2}(t) = \pm\xi(t) + \frac{Tr(A(t))}{2} + \frac{a_{12}'(t)}{2a_{12}(t)} - \frac{\xi'(t)}{2\xi(t)}. \tag{1.4}$$

Introducing the shifted phase function $\varphi(t) = TrA/2 - \theta_1$ we define the functional

$$H(\varphi(t)) = Char(TrA/2-\varphi) = \left(\frac{Tr(A)}{2}\right)^2 - |A| + a_{12}\left(\frac{2\varphi + a_{11} - a_{22}}{2a_{12}}\right)' - \varphi^2. \tag{1.5}$$

Note that the function $\varphi(t)$ is connected with the function $\xi(t) = (\theta_1 - \theta_2)/2$ via transformation

$$\xi(t) = \frac{a_{12}(t)e^{\int_T^t 2\varphi(z)dz}}{2(C - \int_T^t a_{12}(s)e^{\int_T^s 2\varphi(z)dz}ds)}, \quad C = const. \tag{1.6}$$

Theorem 1.1. *Assume $A \in C^1(T,\infty), a_{12} \in C^2(T,\infty)$, $a_{12}(t) \neq 0$ on (T,∞), and there exists a function $\varphi \in C^1(T,\infty)$, such that*

$$\int_T^\infty \left|\frac{H(\varphi(s))}{\xi(s)}\right| e^{\pm 2\int_T^s \Re[\xi(y)]dy} ds < \infty. \tag{1.7}$$

Then the rest state of (1.1) is asymptotically stable if and only if

$$\int_T^\infty \Re[\theta_j(s)ds = -\infty, \quad j = 1,2, \tag{1.8}$$

$$\lim_{t\to\infty} \left|\frac{\theta_j(t,u) - a_{11}(t,u)}{a_{12}(t,u)}\right|(t)e^{\int_T^t \Re[\theta_j(y,u)]dy} = 0, \quad j = 1,2. \tag{1.9}$$

Remark 1.1. *The best choice of the function φ in Theorem 1.1 is such that $H(\varphi(t)) \equiv 0$, which means that the error of approximation is equal to zero and condition (1.7) disappears. It is well known that Riccati equation $H(\varphi(t)) = 0$ can not be solved analytically in general case, so we don't expect to find the best φ in general, but in many cases using theory of asymptotic solutions [7] one can find function φ, such that the function $\frac{H(\varphi)}{\xi}$ is so small that condition (1.7) is satisfied.*

Remark 1.2. *The main function* φ *could be constructed also by using specific asymptotic fundamental matrix solution* Ψ *of (1.1). Indeed (see formula (2.11)) from the given* $\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}$ *the phase function* ξ *may be found from the formula*

$$\xi(t) = \frac{d}{dt} ln \left(\frac{\Psi_{12}(t)}{\Psi_{11}(t)} \right).$$

and the function φ *from (1.6). For example, using Liouville-Green (or WKB) asymptotic solutions we deduce Corollary 1.6 below about asymptotic stability from Theorem 1.1.*

In the case $a_{12} \equiv 0$ Theorem 1.1 is not applicable, but system (1.1) can be solved explicitly, so the following theorem is trivial.

Theorem 1.2. *Assume* $a_{12}(t) \equiv 0, \quad A \in L_1(T, \infty)$*. Then the rest state of (1.1) is asymptotically stable if and only if*

$$\lim_{t\to\infty} \int_T^t a_{jj}(s) ds = -\infty, \quad j = 1, 2, \tag{1.10}$$

$$\lim_{t\to\infty} \left(e^{\int_T^t a_{11}(y)dy} \int_T^t a_{21}(s) e^{\int_t^s (a_{11}-a_{22})(y)dy} ds \right) = 0, \quad j = 1, 2. \tag{1.11}$$

Example 1.1. *Consider the system of linear equations*

$$u_1'(t) = f(t) u_2(t), \quad u_2'(t) = -g(t) u_2(t),$$

$$u_1(t_0) = u_{10}, \quad u_2(t_0) = u_{20},$$

with

$$f(t) = t^{-2a}, \quad g(t) = bt^{2a-2\gamma}, \quad 1 < \gamma < 2a, \quad b \neq 0.$$

For this example asymptotic stability follows from Theorem 1.1.

In the cases when one of the quantities $\pm\Re[\xi(t)]$ is unbounded condition (1.7) is very restrictive. In the next Theorem 1.3 under additional conditions (1.13), (1.14) below we prove the asymptotic stability of the rest state under condition (1.12) less restrictive than (1.7).

Theorem 1.3. *Assume* $a_{12}(t)$ *is not equal to zero for* $t > T$*, and there exists a function* $\varphi \in C^1(T, \infty)$*, such that (1.8) and*

$$\int_T^\infty \left| \frac{H(\varphi(s))}{\xi(s)} \right| ds < \infty. \tag{1.12}$$

$$\Re[\theta_j(t)] \leq 0, \quad j = 1, 2, \tag{1.13}$$

$$\left| \frac{\theta_j(t) - a_{11}(t)}{a_{12}(t)} \right| \leq C, \quad j = 1, 2, \tag{1.14}$$

are satisfied for all $t \geq T$*.*

Then the rest state of (1.1) is asymptotically stable.

Remark 1.3. *If for some positive number p we have*

$$\Re[\xi(t)] \le 0, \quad \Re[\theta_2(t)] \le -p < 0, \quad t > T, \tag{1.15}$$

then condition (1.8), (1.13) of theorem 1.3 may be removed, because they follow from condition (1.15).

Introduce the functions

$$\xi(t) = -\frac{a_{12}(t)}{2\int_T^t a_{12}(s) e^{\int_t^s (2S_{n+1}+a_{22}-a_{11})dy} ds}, \tag{1.16}$$

$$S_0 \equiv 0, \quad S_{n+1}(t) = a_{12}(t)\int_T^t \frac{(S_n^2 - a_{12}a_{21})(s)}{a_{12}(s)} e^{\int_t^s (a_{11}-a_{22})dy} ds, \quad n = 0, 1, \ldots. \tag{1.17}$$

Corollary 1.4. *Assume that the matrix-function $A(t)$ is real valued, and for some $T_1 > T$*

$$a_{12}(t) > 0, \quad t \ge T_1, \tag{1.18}$$

$$\int_{T_1}^{\infty} \left| \frac{S_{n+1}^2(t) - S_n^2(t)}{\xi(t)} \right| dt < \infty, \quad \textit{for some } n, \tag{1.19}$$

$$\int_{T_1}^{\infty} (2\xi + S_{n+1} - a_{11})(t) dt = \infty, \tag{1.20}$$

$$2\xi(t) + S_{n+1}(t) - a_{11}(t) \ge 0 \quad t > T_1, \tag{1.21}$$

$$\left| \frac{S_{n+1}}{a_{12}} \right| (t) \le const, \quad t \ge T_1. \tag{1.22}$$

Then (1.1) is asymptotically stable.

Example 1.2. *Consider the linear system (1.1) with*

$$A(t) = \begin{pmatrix} 0 & 1 \\ -g(t) - 2f'(t) & -2f(t) \end{pmatrix}.$$

Denote

$$S_1(t) = \int_T^t (g(s) + 2f'(s)) e^{\int_t^s 2f(y)dy} ds.$$

If $f \in C^1(T, \infty)$ and for some numbers g_0, f_0

$$0 < f_0 \le f(t), \quad 0 \le g(t) + 2f'(t) \le g_0 < 2f_0^2, \quad t \ge T \tag{1.23}$$

$$\int_{T_1}^{\infty} S_1(t) dt = \infty, \tag{1.24}$$

$$S_1(t) + 2\xi(t) \ge 0, \quad \int_{T_1}^{\infty} \left| \frac{S_1^2(s)}{\xi(s)} \right| ds < \infty, \tag{1.25}$$

then the problem (1.1) is asymptotically stable because all conditions of Corollary 1.4 are satisfied.

Note that it is well known [19] that in the large damping case (1.23) Wintner-Smith condition (1.24) is necessary and sufficient condition of asymptotic stability. So it is possible to get rid of extra conditions (1.25), but we don't know if it could be done in this approach.

Corollary 1.5. *Assume that the matrix-function $A(t)$ is real valued, $a_{12}(t) > 0$ on (T,∞), for all $t \in (T,\infty)$, and*

$$\int_T^\infty \int_T^t \left| a_{21}(s) e^{\int_t^s (a_{22}(y)-a_{11}(y))dy} \right| ds dt < \infty, \tag{1.26}$$

$$\int_T^\infty \left(a_{11}(s) + \frac{a_{12}(s)}{\int_T^t a_{12}(s) e^{\int_t^s (a_{22}-a_{11})(y)dy} ds} \right) dt = -\infty, \tag{1.27}$$

$$a_{11}(t) + \frac{a_{12}(t)}{\int_T^t a_{12}(s) e^{\int_t^s (a_{22}(y)-a_{11}(y)dy} ds} \le 0. \tag{1.28}$$

Then (1.1) is asymptotically stable.

Corollary 1.6. *Assume $a_{12}(t) \neq 0$ on (T,∞), $\quad A \in C^2(T,\infty)$, $a_{22} - a_{11}, a_{12} \in C^3(T,\infty)$, and*

$$\int_T^\infty |k'(t) + k^2(t)\xi(t)| e^{\pm 2 \int_T^t \Re[\xi(s)] ds} dt < \infty, \tag{1.29}$$

where

$$\xi(t) = \sqrt{\left(\frac{TrA}{2}\right)^2 - |A| + a_{12}\left(\frac{a_{11}-a_{22}}{2a_{12}}\right)'}, \quad k(t) = \frac{a_{12}}{2\xi^2}\left(\frac{\xi}{a_{12}}\right)'. \tag{1.30}$$

Then (1.1) is asymptotically stable if and only if (1.8),(1.9) are satisfied.

Corollary 1.7. *Assume $a_{22} - a_{11}, a_{12} \in C^3(T,\infty), A \in C^2(T,\infty)$, $\quad a_{12}(t) \neq 0$ on (T,∞), (1.8),(1.13), (1.14) and*

$$\int_T^\infty |k'(t) + k^2(t)\xi(t)| dt < \infty \tag{1.31}$$

are satisfied. Here functions ξ, θ_j, k are defined in (1.30), (1.4).

Then (1.1) is asymptotically stable.

Note that condition (1.31) is close to the main assumption of asymptotic stability theorems in Pucci and Serrin [16, 17], that $k(t)$ is the function of bounded variation ($\int_T^\infty |k'(t)| dt) < \infty$.

Example 1.3. *Consider system (1.1) with*

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & -2f(t) \end{pmatrix}, \tag{1.32}$$

where

$$f(t) = t^\alpha + it^\beta. \tag{1.33}$$

For the small damping case:

$$-1 < \alpha < 0, \quad \beta < 0, \quad \alpha + \beta < -1 \tag{1.34}$$

conditions of Corollary 1.6 are satisfied and this system is asymptotically stable.

From Corollary 1.7 it follows that this system is asymptotically stable in the more general case:

$$-1 < \alpha < 1, \quad \beta \le \frac{\alpha+1}{2}. \tag{1.35}$$

Here we consider another application of our approach in optical physics. We deduce the transition probability formula for dynamic system described by a general system (1.1) with antihermitian matrix function $A(t)$ and initial conditions:

$$u_1(0) = 1, \quad u_2(0) = 0. \tag{1.36}$$

The associated probability for an atom initially in state a to make a transition after excitation for a time t to state b is

$$P(t) = |u_1(t)|^2$$

Note that if the matrix A in equation (1.1) is antihermitian: $A^* = -A$, then the normalization of the wave function is constant at all times:

$$|u_1(t)|^2 + |u_2(t)|^2 = 1.$$

Introducing the auxiliary functions

$$g(t) = g(0) - 2\int_0^t a_{12}(t) e^{\int_0^t 2\varphi(y)dy}, \tag{1.37}$$

$$\alpha(t) = \frac{1}{2}\Re \ln\left(\frac{g(0)(a_{11}(0)-\theta_2(0))}{g(t)(\theta_1(0)-a_{11}(0))}\right), \quad \beta(t) = \frac{1}{2}\Im \ln\left(\frac{g(0)(a_{11}(0)-\theta_2(0))}{g(t)(\theta_1(0)-a_{11}(0))}\right), \tag{1.38}$$

$$B(t) = \frac{|g(t)g(0)(a_{11}(0)-\theta_2(0))(a_{11}(0)-\theta_1(0))|}{|a_{12}(0)|^2 e^{-2\int_0^t \Re(\theta_1)dy}} \tag{1.39}$$

we have general transition probability formulas

$$|u_1(t)|^2 = B(t)\left(sinh^2\alpha(t) + cos^2\beta(t)\right), \quad |u_2(t)|^2 = 1 - |u_1(t)|^2. \tag{1.40}$$

Note that in view of (1.4),(1.6), (1.37)-(1.40) to calculate transition probability we need to know only the function $\varphi(t)$.

Formula (1.40) allows quickly calculate transition probability for any approximation given via a function φ. Anyway the best choice of φ is such that minimizes $H(\varphi)/\xi$.

Example 1.4. *Consider the dynamic system (1.1) which describes an interaction of two-level atom in the external monochromatic electromagnetic field with frequency* ω*:*

$$u'(t) = \begin{pmatrix} 0 & iWe^{itE}\cos(t\omega) \\ iWe^{-itE}\cos(t\omega) & 0 \end{pmatrix} u(t). \tag{1.41}$$

where E is the difference of energy levels of the atom.

From (1.5) we have

$$\begin{aligned} H(\varphi) &= -\varphi^2 - W^2\cos^2(t\omega) + e^{itE}\cos(t\omega)\left(\frac{\varphi e^{-itE}}{\cos(t\omega)}\right)' \\ &= \varphi' + \varphi(-iE + \omega\tan(t\omega)) - W^2\cos^2(t\omega) - \varphi^2. \end{aligned} \tag{1.42}$$

The function

$$\varphi = a_{12} = iWe^{itE}\cos(t\omega) \tag{1.43}$$

gives a good approximation from a mathematical point of view. From (1.40) we get

$$|u_1(\tau)|^2 = \sin^2\left(W\int_0^t \cos(s\omega)\cos(sE)ds\right) + \sinh^2\left(W\int_0^t \cos(s\omega)\sin(sE)ds\right). \tag{1.44}$$

Note that using expression (1.42) we get

$$\frac{H(\varphi(t))}{\xi(t)} = \frac{W^2\cos^2(t\omega)(e^{2itE}-1)}{-iWe^{itE}\cos(t\omega)} = -2W\sin(tE)\cos(t\omega),$$

or

$$\int_0^t \frac{|H(\varphi(s))|ds}{|\xi(s)|} \le 2W\int_0^t |\sin(sE)\cos(s\omega)|ds \le tW, \tag{1.45}$$

which is small for small tW.

Using rotating wave approximation from optical physics (see[1, 18]) we get another function φ:

$$\varphi_0 = \frac{i}{2}(\omega - E + \Delta), \quad \Delta = \sqrt{(\omega-E)^2 + W^2}. \tag{1.46}$$

From (1.40)

$$|u_1(\tau)|^2 = B\sin^2\left(\frac{t\Delta + t\omega - \eta(t)}{2}\right) + B\sinh^2\left(\frac{1}{2}\ln\frac{R}{(\Delta+\omega)(m-1)}\right). \tag{1.47}$$

where

$$\eta(t) = \tan^{-1}\left(\frac{\omega\tan(t\omega)}{\Delta+\omega}\right), \quad R = \sqrt{(\Delta+\omega)^2\cos^2(t\omega) + \omega^2\sin^2(t\omega)}.$$

$$B = \frac{(\Delta+\omega-E)^2(m-1)(\Delta+\omega)\sqrt{(\Delta+\omega)^2+\omega^2}}{\Delta^2(\Delta+2\omega)^2}, \quad m = \frac{2\Delta(\Delta+2\omega)}{(\Delta+\omega)(\Delta+\omega-E)}. \tag{1.48}$$

If $\omega = 0$, then from (1.47) we get

$$|u_1(\tau)|^2 = \frac{W^2}{E^2+W^2}\left[\sin^2\left(\frac{t\Delta}{2}\right) + \sinh^2\left(\frac{1}{2}ln\frac{\Delta-E}{\Delta+E}\right)\right].$$

If $E = \omega = 0$, then

$$|u_1(\tau)|^2 = \sin^2(tW/2),$$

which is often referred as the Rabi formula [1]. Note that one can estimate the error function for each approximation by using Theorem 2.1 below.

2. Levinson Theorem for 2×2 System and Proofs of Main Results

Suppose we can find the exact solutions of the system

$$\psi'(t) = B(t)\psi(t), \quad t > T, \tag{2.1}$$

with the matrix-function

$$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$$

close to the matrix-function A, which means that the condition (2.7) below is satisfied. Let $\Psi(t)$ is the 2×2 fundamental matrix function of the auxiliary system (2.1). Then the solutions of (2.1) can be represented in the form

$$u(t) = \Psi(t)(C + \varepsilon(t)), \tag{2.2}$$

where $u(t), \varepsilon(t), C$ are the 2-vector columns: $u(t) = \text{colomn}(u_1(t), u_2(t))$, $\varepsilon(t) = \text{colomn}(\varepsilon_1(t), \varepsilon_2(t))$, $C = \text{colomn}(C_1, C_2), C_k$ are arbitrary constants. We can consider representation (2.2) as a definition of the error vector-function $\varepsilon(t)$.

Following theorem is a version of the Levinson Theorem [15, 7] about asymptotic solutions for 2×2 systems:

Theorem 2.1. *Assume there exist a function $\xi \in C^1[T, \infty)$ such that*

$$L(t) \equiv \max_{j=1,2} \left| \frac{Char(\theta_j(t))}{2\xi(t)} e^{(-1)^{j-1} \int_T^t 2\Re[\xi(y)]dy} \right| \in L^1(T, \infty). \tag{2.3}$$

Then every solution of (1.1) can be represented in form (2.2) and the error vector-function $\varepsilon(t)$ can be estimated as

$$\|\varepsilon(t)\| \le \|C\| \Big(-1 + \exp \int_t^\infty |L(s)| ds \Big), \tag{2.4}$$

where C is the constant vector and $\|\cdot\|$ is the Euclidean vector (or matrix) norm: $\|\varepsilon(t)\| = \sqrt{\varepsilon_1^2(t) + \varepsilon_2^2(t)}$.

Remark 2.1. *From (2.3),(2.4) it follows that $\varepsilon(t) = o(1), \quad t \to \infty$. Also if $Char(\theta_1) = Char(\theta_2) \equiv 0$, then $\varepsilon(t) \equiv 0$.*

Remark 2.2. *Trying to find asymptotic solutions that are minimizing the error or corresponding function H given by formula (1.5), one can choose the function φ for example by the formula (see also (1.30))*

$$\varphi^2(t) = \left(\frac{TrA}{2} \right)^2 - |A| + a_{12} \left(\frac{a_{11} - a_{22}}{2a_{12}} \right)'.$$

Then asymptotic solutions obtained by this choice via formulas (1.4) will coincide with the well known Liouville-Green functions. Another choice of φ is given in (2.30) below.

Proof of Theorem 2.1. The substitution $u(t) = \Psi(t)v(t)$ transforms (1.1) into

$$v'(t) = M(t)v(t), \quad M(t) = \Psi^{-1}(A\Psi - \Psi')(t).$$

By integration we get

$$v(t) = C - \int_t^b M(s)v(s)ds, \quad T < t < b, \tag{2.5}$$

where the constant vector C is chosen as in (2.2).

Estimating $v(t)$

$$\|v(t)\| \le \|C\| + \int_t^b \|M(s)\|\|v(s)\|ds,$$

and using Gronwall's inequality we have

$$\|v(t)\| \le \|C\|e^{\int_t^b \|M(s)\|ds}.$$

From representation (2.2) we have

$$\varepsilon(t) = \Psi^{-1}u - C = v - C = -\int_t^b M(s)v(s)ds,$$

and using previous estimate we get

$$\begin{aligned}\|\varepsilon(t)\| &\le \int_t^b \|Mv\|ds \\ &\le \|C\| \int_t^b \|M(s)\| \exp\Big(\int_s^b \|M\|dy\Big)ds \\ &= \|C\|\Big(-1 + \exp\big(\int_t^b \|M\|ds\big)\Big),\end{aligned}$$

or

$$\|\varepsilon(t)\| \le \|C\|\Big(-1 + \exp\big(\int_t^\infty \|\Psi^{-1}(A\Psi - \Psi')(s)\|ds\big)\Big). \tag{2.6}$$

Note that error function $\varepsilon(t)$ is bounded if

$$\int_t^\infty \|\Psi^{-1}(A\Psi - \Psi')(s)\|ds < \infty. \tag{2.7}$$

To finish the proof we should calculate matrix function M in terms of characteristic functions $Char(\theta_j)$ using the construction of approximate fundamental matrix solution of (1.1).

To construct the approximate fundamental matrix function Ψ let us seek approximate solutions of (1.1)

$$u_1' = a_{11}u_1 + a_{12}u_2, \quad u_2' = a_{21}u_1 + a_{22}u_2,$$

as a linear combination of exponential functions

$$u_1 = C_1 e^{\int_T^t \theta_1(y)dy} + C_2 e^{\int_T^t \theta_2(y)dy}. \tag{2.8}$$

Substituting this representation for u_1 in the first equation

$$a_{12}u_2 = u_1' - a_{11}u_1 = (\theta_1 - a_{11})C_1 e^{\int_1^t \theta_1(y)dy} + C_2(\theta_2 - a_{11})e^{\int_T^t \theta_2(y)dy},$$

and solving for u_2 we have

$$u_2(t) = U_1(t)C_1 e^{\int_T^t \theta_1(y)dy} + U_2(t)C_2 e^{\int_T^t \theta_2(y)dy},$$

$$U_1(t) = \frac{\theta_1 - a_{11}}{a_{12}}, \quad U_2(t) = \frac{\theta_2 - a_{11}}{a_{12}}, \quad U_1(t) - U_2(t) = \frac{2\xi(t)}{a_{12}(t)}, \tag{2.9}$$

or

$$u(t) = \Psi(t)C, \tag{2.10}$$

where the fundamental matrix $\Psi(t)$ is defined by the formula

$$\Psi(t) = \begin{pmatrix} 1 & 1 \\ U_1(t) & U_2(t) \end{pmatrix} \begin{pmatrix} e^{\int_T^t \theta_1(y)dy} & 0 \\ 0 & e^{\int_T^t \theta_2(y)dy} \end{pmatrix}. \tag{2.11}$$

Define

$$\xi(t) = \frac{\theta_1(t) - \theta_2(t)}{2}, \quad Char_j(t) = Char(\theta_j(t)). \tag{2.12}$$

If $A \in C^1(T,\infty)$, $a_{12} \in C^2(T,\infty)$, $a_{12}(t)$ is not equal to zero on (T,∞), then following formulas are true

$$|\Psi(t)| = det[\Psi(t)] = -\frac{2\xi(t)}{a_{12}(t)} e^{\int_T^t (\theta_1+\theta_2)dy}, \tag{2.13}$$

$$\frac{Char_2(t) - Char_1(t)}{2\xi} = \theta_1 + \theta_2 - Tr(A) + \frac{\xi'}{\xi} - \frac{a_{12}'}{a_{12}} \tag{2.14}$$

$$\Psi M \Psi^{-1} = A - \Psi'\Psi^{-1} = \frac{1}{2\xi} \begin{pmatrix} 0 & 0 \\ U_1 Char_2 - U_2 Char_1 & Char_1 - Char_2 \end{pmatrix}, \tag{2.15}$$

$$M(t) = \Psi^{-1}A\Psi - \Psi^{-1}\Psi'(t) = \frac{1}{2\xi(t)} \begin{pmatrix} Char_1(t) & e^{-2\int_T^t \xi dy} Char_2(t) \\ -e^{2\int_T^t \xi dy} Char_1(t) & -Char_2(t) \end{pmatrix}. \tag{2.16}$$

From Liouville's formula

$$\frac{|\Psi|'}{|\Psi|} = Tr(A) = a_{11} + a_{22} \tag{2.17}$$

in view of (2.11) we have

$$\frac{(\theta_1 - \theta_2)'}{\theta_1 - \theta_2} - \frac{a_{12}'}{a_{12}} + \theta_1 + \theta_2 - a_{11} - a_{22} = 0$$

or another version of Liouville's formula

$$\theta_1 + \theta_2 = Tr(A) - \frac{\xi'}{\xi} + \frac{a_{12}'}{a_{12}}. \tag{2.18}$$

It easy to check that the functions θ_j from (1.4) satisfy (2.18).

Remark 2.3. *From (1.3) and (2.18) it follows that*

$$Char_1(t) = Char_2(t) \equiv H(t), \tag{2.19}$$

and formula (2.16) turns to

$$M(t) = \Psi^{-1}A\Psi - \Psi^{-1}\Psi' = \frac{H(t)}{2\xi(t)} \begin{pmatrix} -1 & -e^{-2\int_T^t \xi(y,u(y))dy} \\ e^{2\int_T^t \xi(y,u(y))dy} & 1 \end{pmatrix}. \tag{2.20}$$

Formulas (2.13)-(2.16) can be checked by direct calculations. Indeed,

$$Char_2(t) - Char_1(t) = (\theta_1 - \theta_2)\left(\theta_1 + \theta_2 - Tr(A) + \frac{W[a_{12}, \theta_1 - \theta_2]}{a_{12}(\theta_1 - \theta_2)}\right).$$

From (2.11)

$$\Psi^{-1}(t) = \frac{1}{U_2 - U_1}\begin{pmatrix} e^{-\int_T^t \theta_1 dy} & 0 \\ 0 & e^{-\int_T^t \theta_2 dy}\end{pmatrix}\begin{pmatrix} U_2 & -1 \\ -U_1 & 1\end{pmatrix}$$

$$\Psi'(t) = \begin{pmatrix} \theta_1 & \theta_2 \\ \Lambda_1 U_1 & \Lambda_2 U_2\end{pmatrix}\begin{pmatrix} e^{\int_T^t \theta_1 dy} & 0 \\ 0 & e^{\int_T^t \theta_2 dy}\end{pmatrix}$$

where

$$\Lambda_j = \theta_j + \frac{(\theta_j - a_{11})'}{\theta_j - a_{11}} - \frac{a_{12}'}{a_{12}} = \theta_j + \frac{W[a_{12}, \theta_j - a_{11}]}{a_{12}(\theta_j - a_{11})}, \quad j = 1, 2.$$

So

$$\Psi'\Psi^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} + \frac{Char_1 U_2 - Char_2 U_1}{2\xi} & a_{22} + \frac{Char_2 - Char_1}{2\xi}\end{pmatrix}$$

Indeed,

$$\Psi'\Psi^{-1} = \frac{1}{U_2 - U_1}\begin{pmatrix} \theta_1 & \theta_2 \\ \Lambda_1 U_1 & \Lambda_2 U_2\end{pmatrix}\begin{pmatrix} U_2 & -1 \\ -U_1 & 1\end{pmatrix} =$$

$$\frac{1}{U_2 - U_1}\begin{pmatrix} \theta_1 U_2 - \theta_2 U_1 & \theta_2 - \theta_1 \\ (\Lambda_1 - \Lambda_2)U_1 U_2 & \Lambda_2 U_2 - \Lambda_1 U_1\end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} + \frac{Char_2 U_1 - Char_1 U_2}{2\xi} & a_{22} + \frac{Char_2 - Char_1}{2\xi}\end{pmatrix},$$

in view of

$$\frac{\theta_1 U_2 - \theta_2 U_1}{U_2 - U_1} = \theta_1 + \frac{(\theta_1 - \theta_2)U_1}{U_2 - U_1} = \theta_1 + a_{11} - \theta_1 = a_{11},$$

and

$$\frac{\Lambda_2 U_2 - \Lambda_1 U_1}{U_2 - U_1} = \frac{(\Lambda_2 - \Lambda_1)U_1}{U_2 - U_1} + \Lambda_2 = \frac{(a_{11} - \theta_1)(\Lambda_2 - \Lambda_1)}{2\xi} + \Lambda_2$$

$$= \frac{a_{11} - \theta_1}{2\xi}\left[-2\xi + \frac{(\theta_2 - a_{11})'}{\theta_2 - a_{11}} - \frac{(\theta_1 - a_{11})'}{\theta_1 - a_{11}}\right] + \theta_2 + \frac{(\theta_2 - a_{11})'}{\theta_2 - a_{11}} - \frac{a_{12}'}{a_{12}} =$$

$$\theta_1 - a_{11} + \frac{(\theta_2 - a_{11})'}{\theta_2 - a_{11}} \left(\frac{a_{11} - \theta_1}{2\xi} + 1 \right) + \frac{(\theta_1 - a_{11})'}{2\xi} + \theta_2 - \frac{a'_{12}}{a_{12}} =$$

$$= \theta_1 + \theta_2 - a_{11} + \frac{\xi'}{\xi} - \frac{a'_{12}}{a_{12}} = a_{22} + \frac{Char_2 - Char_1}{2\xi},$$

and

$$\frac{U_1 U_2 (\Lambda_1 - \Lambda_2)}{U_2 - U_1} = \frac{(\theta_1 - a_{11})(\theta_2 - a_{11})}{-2\xi a_{12}} \left[2\xi + \frac{(\theta_1 - a_{11})'}{\theta_1 - a_{11}} - \frac{(\theta_2 - a_{11})'}{\theta_2 - a_{11}} \right]$$

$$= -\frac{(\theta_1 - a_{11})(\theta_2 - a_{11})}{a_{12}} + \frac{(\theta_2 - a_{11})'(\theta_1 - a_{11}) - (\theta_1 - a_{11})'(\theta_2 - a_{11})}{2a_{12}\xi} =$$

$$= a_{21} + \frac{Char_1 U_2 - Char_2 U_1}{2\xi} = a_{21} - \frac{Char_1}{a_{12}} + \frac{(Char_1 - Char_2)(\theta_1 - a_{11})}{2a_{12}\xi},$$

where we use the calculations

$$\frac{Char_2 U_1 - Char_1 U_2}{2\xi} =$$

$$\frac{\theta_2 - a_{11}}{2a_{12}\xi} \left[|A| + \theta_1^2 - \theta_1(a_{11} + a_{22}) + (\theta_1 - a_{11})' - \frac{a'_{12}(\theta_1 - a_{11})}{a_{12}} \right] +$$

$$-\frac{\theta_1 - a_{11}}{2a_{12}\xi} \left[|A| + \theta_2^2 - \theta_2(a_{11} + a_{22}) + (\theta_2 - a_{11})' - \frac{a'_{12}(\theta_2 - a_{11})}{a_{12}} \right] =$$

$$= \frac{|A|}{-a_{12}} + \frac{\theta_1\theta_2 - a_{11}(\theta_1 + \theta_2)}{a_{12}} + \frac{a_{11} + a_{22}}{2a_{12}\xi} [\theta_2(\theta_1 - a_{11}) - \theta_1(\theta_2 - a_{11})] +$$

$$+ \frac{(\theta_2 - a_{11})(\theta_1 - a_{11})' - (\theta_1 - a_{11})(\theta_2 - a_{11})'}{2a_{12}\xi} =$$

$$a_{21} + \frac{(\theta_1 - a_{11})(\theta_2 - a_{11})}{a_{12}} + \frac{W[\theta_2 - a_{11}, \theta_1 - a_{11}]}{2a_{12}\xi}.$$

The final estimate (2.4) follows from (2.6) and (2.16). □

Proof of Theorem 1.1. From condition (1.7) of Theorem 1.1 and formula (2.20) it follows that

$$\|M(t)\| \in L_1(T, \infty),$$

and condition (2.3) of Theorem 2.1 is satisfied. Applying Theorem 2.1 we obtain representation (2.2) for solutions of (1.1). From (2.2) and (2.4) we get stability inequality

$$\|u(t)\| \leq c \cdot \|\Psi(t)C\|. \tag{2.21}$$

Because of this estimate all solutions of (1.1) are stable and attractive if and only if

$$\lim_{t \to \infty} (\|\Psi(t)\|) = 0.$$

This condition is satisfied because of conditions (1.8),(1.9) of Theorem 1.1 and formula (2.11). □

Proof of Example 1.1. We have

$$A = \begin{pmatrix} 0 & f(t) \\ -g(t) & 0 \end{pmatrix},$$

and

$$a_{12} = f(t), \quad Tr(A) = 0, \quad |A| = f(t)g(t), \quad H(t) = -fg - F^2 + f(t)\left(\frac{F(t)}{f(t)}\right)'.$$

Choosing

$$\xi = \frac{ib}{t^\gamma}$$

we get

$$k = O(t^{\gamma-1}), \quad \frac{H(t)}{\xi} = -\frac{fg}{\xi} - \xi(1-k^2) + k' = O(t^{\gamma-2}) \in L_1(T,\infty), \quad \text{if} \quad \gamma > 1,$$

and condition (1.7) is satisfied. From

$$\theta_{1,2} = \xi + \frac{a_{12}'}{2a_{12}} - \frac{\xi'}{2\xi} = \pm\frac{ib}{t^\gamma} + \frac{\gamma - 2a}{2t}$$

$$\Re[\theta_j] = \frac{\gamma - 2a}{2t} < 0, \quad \text{if} \quad \gamma < 2a,$$

it follows that condition (1.8) is satisfied.

If $\gamma > 1$ then condition (1.9) is satisfied as well:

$$\frac{\theta_j - a_{11}}{a_{12}} e^{\int_1^t \Re[\theta_j]dy} = t^{2a}\left(\frac{ib}{t^\gamma} + \frac{\gamma-2a}{2t}\right)\exp\left(\int^t \frac{(\gamma-2a)dy}{y}\right) = O(t^{\gamma-1}) \to 0,$$

when $t \to \infty$ □

Denote by $G(t,s) = \Psi(t)\Psi^{-1}(s)$ the Cauchy matrix function of (1.1).

Lemma 2.2. *Assume that conditions (1.13), (1.14) are satisfied. Then*

$$|G(t,s)| = |\Psi(t)\Psi^{-1}(s)| \le C\left|\frac{a_{12}(s)}{\xi(s)}\right|, \quad T \le s \le t, \tag{2.22}$$

$$\|\Psi(t)M(t)\Psi^{-1}(t)\| \le C\left|\frac{H(t)}{a_{12}}\right|, \quad t \ge T. \tag{2.23}$$

Proof of Lemma 2.2. From condition (1.14) it follows that

$$|U_j(t)| \le C, \quad j = 1,2, \quad \text{for all} \quad t \ge T.$$

By direct calculations

$$G(t,s) = \frac{1}{U_2(s) - U_1(s)} \begin{pmatrix} e^{\int_s^t \theta_1 dy} & e^{\int_s^t \theta_2 dy} \\ U_1(t)e^{\int_s^t \theta_1 dy} & U_2(t)e^{\int_s^t \theta_2 dy} \end{pmatrix} \begin{pmatrix} U_2(s) & -1 \\ -U_1(s) & 1 \end{pmatrix}$$

So estimate (2.22) follows from

$$|G_{kj}(t,s)| \le \frac{C}{|U_2(s) - U_1(s)|} = C\left|\frac{a_{12}(s)}{2\xi(s)}\right|, \quad k,j = 1,2.$$

The estimate (2.23) follows from the formula (2.15). □

Proof of Theorem 1.2. Proof follows directly from the explicit formula for fundamental matrix function in the case $a_{12} \equiv 0$:

$$\Psi(t) = \begin{pmatrix} e^{\int_T^t a_{11}(y)dy}, & 0 \\ e^{\int_T^t a_{22}(y)dy}[C + \int_T^t a_{21}(s)e^{\int_T^s (a_{11}-a_{22})(y)dy}ds], & e^{\int_T^t a_{22}(y)dy} \end{pmatrix}.$$

□

Proof of Theorem 1.3. Consider the system (1.1). By substitution

$$u(t) = \Psi(t)v(t),$$

we get

$$v'(t) = M(t)v(t), \quad v(t) = C + \int_T^t M(s)v(s)ds,$$

or

$$\Psi^{-1}(t)u(t) = C + \int_T^t M(s)\Psi^{-1}(s)u(s)ds, \quad T \le s \le t, \tag{2.24}$$

$$u(t) = \Psi(t)C + \int_T^t G(t,s)\Psi(s)M(s)\Psi^{-1}(s)u(s)ds. \tag{2.25}$$

From this representation and Lemma 2.2 we obtain the estimates

$$\|u(t)\| \le \|\Psi(t)C\| + \int_T^t \|G(t,s)\| \cdot \|\Psi(s)M(s)\Psi^{-1}(s)u(s)\|ds$$

$$\le \|\Psi(t)C\| + \int_T^t \left|\frac{H(s)}{\xi(s)}\right| \|u(s)\|ds.$$

Applying Gronwall's inequality (see for example [10]) we get

$$\|u(t)\| \le \|\Psi(t)C\| + \int_T^t \|\Psi(s)C\| \left|\frac{H(s)}{\xi(s)}\right| \exp\left(\int_T^s \left|\frac{H(y)}{\xi(y)}\right| dy\right) ds,$$

$$\|u(t)\| \le \|\Psi(t)C\|_\infty \left(1 + \int_T^t \left|\frac{H(s)}{\xi(s)}\right| \exp\left(\int_T^s \left|\frac{H(y)}{\xi(y)}\right| dy\right) ds\right),$$

where $\|u(t)\|_\infty = sup_{t \ge T}\|u(t)\|$.

So we obtain the stability estimate

$$\|u(t)\| \le \|\Psi(t)C\|_\infty \exp\left(\int_T^t \left|\frac{H(s)}{\xi(s)}\right| ds\right). \tag{2.26}$$

Using this inequality we can estimate (2.25) again

$$\|u(t) - \Psi(t)C\| \le \|\Psi(t)C\|_\infty \left(\exp \int_T^t \left|\frac{H(s)}{\xi(s)}\right| ds - 1\right). \tag{2.27}$$

From conditions (1.13),(1.14) of Theorem 1.3 and formula (2.11) we have

$$\|\Psi(t)C\| \le const.$$

So from stability inequality (2.26) and condition (1.12) of Theorem 1.3 we get stability of (1.1).

From (1.8),(1.9) we have

$$\lim_{t\to\infty}\|\Psi(t,u)C\| = 0,$$

and asymptotic stability of (1.1) follows from the estimate (2.26).

□

Proof of Corollary 1.4. We deduce Corollary 1.4 from Theorem 1.3 by choosing ξ as in (1.16), and

$$\varphi = S_{n+1} + \frac{a_{22}-a_{11}}{2}. \tag{2.28}$$

From the condition $a_{12} > 0$ it follows that for all $t \ge T_1 > T$

$$\int_T^t a_{12} e^{\int_t^s (2S_{n+1}+a_{22}-a_{11})dy} \ge \int_T^{T_1} a_{12} e^{\int_t^s (2S_{n+1}+a_{22}-a_{11})dy} = \gamma_1(T_1) > 0,$$

and from (1.16)

$$-\frac{a_{12}}{2\gamma_1(T_1)} \le \xi < 0, \quad t \ge T_1 > T. \tag{2.29}$$

By direct calculations we get from (1.5),(2.28)

$$H(t) = \left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21} + a_{12}\left(\frac{2\varphi+a_{11}-a_{22}}{2a_{12}}\right)' - \varphi^2 =$$

$$\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21} + a_{12}\left(\frac{S_{n+1}}{a_{12}}\right)' - (S_{n+1} + \frac{a_{22}-a_{11}}{2})^2 =$$

$$a_{12}a_{21} + a_{12}\left(\frac{S_{n+1}}{a_{12}}\right)' + (a_{11}-a_{22})S_{n+1} - S_{n+1}^2 = S_n^2 - S_{n+1}^2,$$

if S_{n+1} are the solutions of first order equations:

$$S_0 \equiv 0, \quad \left(\frac{S_{n+1}}{a_{12}}\right)' + (a_{11}-a_{22})\frac{S_{n+1}}{a_{12}} = \frac{S_n^2}{a_{12}} - a_{21}, \quad n = 0,1,2,\ldots$$

and given by formulas (1.17). So condition (1.12) of Theorem 1.3 turns to (1.19).

In view of (1.6):

$$\frac{\xi'}{2\xi} - \frac{a'_{12}}{2a_{12}} = \varphi + \xi$$

we have from (1.21),(2.28)

$$\theta_1 = \xi + \frac{TrA}{2} + \frac{a'_{12}}{2a_{12}} - \frac{\xi'}{2\xi} = \frac{TrA}{2} - \varphi = a_{11} - S_{n+1} \le a_{11} - S_{n+1} - 2\xi \le 0$$

$$\theta_2 = \theta_1 - 2\xi = a_{11} - S_{n+1} - 2\xi \le 0.$$

From condition (1.20) it follows condition (1.8):

$$\int^t \theta_j(s)ds \to -\infty, \quad j = 1,2, \quad t \to \infty.$$

Finally condition (1.14) of Theorem 1.3 follows from (2.29) and (1.22):

$$\left|\frac{2\xi}{a_{12}}\right| = \frac{1}{\int_T^t a_{12} e^{\int_t^s (2S_{n+1}+a_{22}-a_{11})dy}} \leq \frac{1}{\gamma_1}, \quad \left|\frac{\theta_1 - a_{11}}{a_{12}}\right| = \left|-\frac{S_{n+1}}{a_{12}}\right| \leq const.$$

□

Proof of Example 1.2. From (1.16),(1.17) with $n = 0$ we have

$$\xi(t) = -\frac{1}{2\int_T^t e^{\int_t^s 2(S_1-f)dy}}, \quad S_1(t) = \int_T^t (g+2f')e^{\int_t^s 2f dy} ds,$$

and conditions (1.19), (1.21) of Corollary 1.4 turns to condition (1.25). From the estimate

$$S_1(t) \leq g_0 \int_T^t e^{\int_t^s 2f dy} ds \leq g_0 \int_T^t e^{2f_0(s-t)} ds \leq \frac{g_0}{2f_0}$$

condition (1.22) is fulfilled. The condition (1.20) follows from the estimates

$$\int_T^t e^{\int_T^s 2(S_1-f)dy} ds \leq \int_T^\infty e^{(s-T)(\frac{g_0}{2f_0}-f_0)} ds = \frac{1}{f_0 - \frac{g_0}{2f_0}} \equiv \frac{1}{f_1} < \infty$$

$$\int_{T_1}^t 2\xi(\tau)d\tau = -\int_{T_1}^t \left(\ln \int_T^\tau e^{\int_T^s 2(S_1-f)dy} ds\right)'(\tau)d\tau =$$

$$\ln\left(\int_T^{T_1} e^{\int_T^s 2(S_1-f)dy}\right) - \ln\left(\int_T^t e^{\int_T^s 2(S_1-f)dy}\right) \geq \ln(f_2(T,T_1)) + \ln(f_1).$$

So (1.20) follows from (1.24):

$$\int_{T_1}^t (2\xi + S_1)dy \geq \ln(f_1 f_2) + \int_{T_1}^t S_1 dy \to \infty, \quad t \to \infty.$$

□

Proof of Corollary 1.5. Choosing

$$\varphi = \frac{a_{22} - a_{11}}{2} \tag{2.30}$$

we get

$$\xi = -\frac{a_{12}(t)}{2\int_T^t a_{12}(s) e^{\int_t^s (a_{22}-a_{11})dy} ds} \leq 0, \quad \theta_1 = a_{11},$$

$$\theta_2 = a_{11} - 2\xi = a_{11} + \frac{a_{12}}{\int_T^t a_{12}(s) e^{\int_t^s (a_{22}-a_{11})dy} ds},$$

$$H(t) = \left(\frac{Tr(A)}{2}\right)^2 - |A| + a_{12}\left(\frac{2\varphi + a_{11} - a_{22}}{2a_{12}}(t)\right)' - \varphi^2(t) =$$

$$\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}a_{21} - \varphi^2 = a_{12}a_{21},$$

$$\frac{H(t)}{2\xi} = -a_{21}(t)\int_T^t a_{12}(s)e^{\int_t^s (a_{22}-a_{11})dy}ds.$$

So condition (1.12) of Theorem 1.3 turns to (1.26). The rest of the proof is similar to the proof of Corollary 1.4.

□

Proof of Corollary 1.6. From

$$\varphi = \frac{\xi'}{2\xi} - \xi - \frac{a_{12}'}{2a_{12}} = \frac{a_{12}}{2\xi}\left(\frac{\xi}{a_{12}}\right)' - \xi = (k-1)\xi,$$

we get from (1.5)

$$H(t) = \left(\frac{Tr(A)}{2}\right)^2 - |A| + a_{12}\left(\frac{a_{11}-a_{22}}{2a_{12}}\right)' - \xi^2 + k^2\xi^2 + k'\xi.$$

Choosing ξ as in (1.30) we get

$$\frac{H(t)}{\xi(t)} = k'(t) + k^2(t)\xi(t), \tag{2.31}$$

and Corollary 1.6 follows from Theorem 1.1.

□

Proof of Corollary 1.7. Corollary 1.7 follows from Theorem 1.3 by choosing ξ as in (1.30).

□

Proof of Example 1.3. From (1.30)

$$\xi = \sqrt{f^2(t) - 1 + f'(t)}.$$

To check conditions of Corollary 1.6 denote

$$P = \Re[f^2(t) - 1 + f'(t)] = t^{2\alpha} - t^{2\beta} - 1 + \alpha t^{\alpha-1},$$

$$Q = \Im[f^2(t) - 1 + f'(t)] = 2t^{\alpha+\beta} + \beta t^{\beta-1}.$$

From $\alpha < 0, \beta < 0$ we get

$$\sqrt{P^2+Q^2} + P = \frac{Q^2}{\sqrt{P^2+Q^2} - P} = \frac{Q^2}{2}(1+o(1)), \quad t \to \infty$$

and

$$\Re[\xi] = \Re[P + iQ] = \frac{\sqrt{P + \sqrt{P^2+Q^2}}}{\sqrt{2}} = \frac{Q}{2}(1+o(1)).$$

So condition (1.29) follows from

$$|\xi| = O(1), \quad k' + k^2\xi = O(t^{-2}), \quad t \to \infty,$$

and

$$\pm\int_T^\infty \Re[\xi]dt = \frac{1}{2}\int_T^\infty [2t^{\alpha+\beta}+\beta t^{\beta-1}2(1+o(1))]dt < \infty.$$

From $\alpha > -1$ it follows that conditions (1.8) (1.9) are satisfied as well:

$$\Re[\theta_j] = \Re\left[\pm\xi - f(t) - \frac{\xi'}{2\xi}\right] = -t^\alpha(1+o(1)) + O(1/t), \quad t\to\infty.$$

Further we will show that conditions of Corollary 1.7 are satisfied if conditions (1.35) are fulfilled. Denote

$$V(t) = \Re[f-\xi] = t^\alpha - \sqrt{(P+R)/2}, \quad R = \sqrt{P^2+Q^2}.$$

By calculations

$$V(t) = \frac{t^{2\alpha}-(P+R)/2}{t^\alpha+\sqrt{(P+R)/2}} = \frac{2t^{2\alpha}-P-R}{2t^\alpha+\sqrt{2P+2R}} = \frac{K(t)}{(2t^\alpha+\sqrt{2P+2R})(2t^{2\alpha}-P+R)},$$

$$K = (2t^{2\alpha}-P)^2 - R^2 = 4t^{2\alpha}[1-\alpha t^{\alpha-1}-\beta t^{2\beta-1-\alpha}-\beta^2 t^{2\beta-2-\alpha}].$$

From the formulas for K and Q it follows that

$$K = t^{2\alpha}(1+o(1), \quad Q = 2t^{\alpha+\beta}(1+o(1)).$$

To prove

$$\int_T^\infty V(t)dt = \infty$$

we divide the plane (α,β) on 3 regions:

$$\{\alpha\ge\beta, \alpha\ge 0\}, \quad \{\beta\ge\alpha, \beta\ge 0\}, \quad \{\alpha\le 0, \beta\le 0\},$$

and prove it in each region separately.

Region 1: $\alpha\ge\beta, \quad \alpha\ge 0$.

From $P = t^{2\alpha}(1+o(1)), \quad R = t^{2\alpha}(1+o(1)), \quad R-P = \frac{Q^2}{P+R} = t^{2\beta}(1+o(1))$,

$$V(t) = \frac{t^{2\alpha}(1+o(1))}{t^\alpha+\sqrt{(P+R)/2}} = \frac{t^{2\alpha}(1+o(1))}{t^{3\alpha}} = t^{-\alpha}(1+o(1)),$$

and if $\alpha < 1$ then the formula is true.

Region 2: $\beta\ge\alpha, \quad \beta\ge 0$.

From $P = -t^{2\beta}(1+o(1)), \quad R = t^{2\beta}(1+o(1)), \quad R-P = t^{2\beta}, \quad R+P = \frac{Q^2}{R-P} = t^{2\alpha}(1+o(1))$,

$$V(t) = \frac{t^{2\alpha}(1+o(1))}{t^\alpha+\sqrt{(P+R)/2}} = \frac{t^{2\alpha}(1+o(1))}{t^{\alpha+2\beta}} = t^{\alpha-2\beta}(1+o(1)),$$

and if $\alpha+1-2\beta > 0$ then the formula is true.

Region 3: $\alpha\le 0, \quad \beta\le 0$.

From $P=-1+o(1),\quad R=1+o(1),\quad R-P=R=2+o(1),\quad R+P=\frac{Q^2}{R-P}=t^{2\alpha+2\beta}(1+o(1))$,

$$V(t)=\frac{t^{2\alpha}(1+o(1))}{t^{\alpha}+\sqrt{(P+R)/2}}=\frac{t^{2\alpha}(1+o(1))}{t^{\alpha}+t^{\alpha+\beta}}=t^{\alpha}(1+o(1)),$$

and if $\alpha+1>0$ then the formula is true.

Now we are ready to check conditions (1.8):

$$e^{\int_T^t \Re[\theta_1]ds}=e^{\int_T^t \Re[\xi-f-\frac{\xi'}{2\xi}]ds}=$$

$$\left|\frac{\xi(T)}{\xi(t)}\right|^{1/2}e^{-\int_T^t \Re[f-\xi]ds}\le Ce^{-\int_T^t V(s)ds}\to 0,\quad t\to\infty.$$

From $\Re[\xi]\ge 0$ we get

$$e^{\int_T^t \Re[\theta_2]ds}=e^{-\int_T^t \Re[\xi+f-\frac{\xi'}{2\xi}]ds}=\left|\frac{\xi(T)}{\xi(t)}\right|^{1/2}e^{-\int_T^t \Re[f]ds}\le Ce^{-\int_T^t V(s)ds}\to 0,\quad t\to\infty.$$

Conditions (1.13) are obviously true.

To check conditions (1.14) note that $\alpha>0$ or $\beta>0$ then

$$|f(t)|=\sqrt{t^{2\alpha}+t^{2\beta}}\to\infty,\quad t\to\infty,\quad \frac{|f'|}{|f|}\le\frac{C}{t}$$

$$|\xi-f|=\sqrt{f^2+f'-1}-f=\frac{f'-1}{f+\sqrt{f^2+f'-1}}=\frac{f'-1}{f(1+o(1))}\le C\left(\frac{f'}{f}+\frac{1}{f}\right)\le C$$

If $\alpha\le 0$ or $\beta\le 0$ then

$$|\xi-f|\le|\xi|+|f|\le C.$$

From these estimates and $\frac{\xi'}{\xi}=\frac{1+o(1)}{t}$ it follows that conditions (1.14) are satisfied.

At last (1.31) is satisfied in view of:

$$k(t)=\frac{\xi'}{2\xi^2}=\frac{1+o(1)}{t},\quad k'(t)=\frac{1+o(1)}{t^2},\quad t\to\infty.$$

□

Proof of transition probability formula (1.40). From representation (2.10) we get

$$u_1(t)=C_1e^{\int_0^t\theta_1dy}+C_2e^{\int_0^t\theta_2dy}=$$

$$\sqrt{C_1C_2}e^{\int_0^t\frac{\theta_1+\theta_2}{2}dy}\left(\sqrt{\frac{C_1}{C_2}}e^{\int_0^t\frac{\theta_1-\theta_2}{2}dy}+\sqrt{\frac{C_2}{C_1}}e^{\int_0^t\frac{\theta_2-\theta_1}{2}dy}\right).$$

In view of $x+\frac{1}{x}=e^{\ln x}+e^{-\ln x}=2\cosh(\ln x)$, and

$$\theta_1-\theta_2=-\frac{g'(t)}{g(t)},\quad g(t)=g(0)-\int_0^t be^{\int_0^s 2\varphi dy}ds,\quad \frac{\theta_{12}'}{\theta_{12}}=2\varphi+\frac{b'}{b}+\theta_{12}$$

we have

$$u_1(t) = 2\sqrt{C_1C_2}e^{\int_0^t \frac{\theta_1+\theta_2}{2}dy}\cosh\left(\frac{1}{2}\ln\frac{C_1g(0)}{C_2g(t)}\right),$$

and

$$|u_1(t)|^2 = 4|C_1C_2|e^{\int_0^t \Re(\theta_1+\theta_2)dy}\left|\cosh\left(\frac{1}{2}\ln\frac{C_1g(0)}{C_2g(t)}\right)\right|^2.$$

From initial conditions (1.36) we get

$$C_1+C_2 = 1, \quad C_1U_1(0)+C_2U_2(0) = 0,$$

or

$$C_1 = \frac{-U_2(0)}{U_1(0)-U_2(0)} = \frac{a_{11}(0)-\theta_2(0)}{2\xi(0)}, \quad C_2 = \frac{U_1(0)}{U_1(0)-U_2(0)} = \frac{\theta_1(0)-a_{11}(0)}{2\xi(0)}.$$

Further we have

$$|u_1(t)|^2 = 4|C_1C_2|e^{\int_0^t \Re(\theta_1+\theta_2)dy}\left|\cosh\left(\frac{1}{2}\ln\frac{C_1g(0)}{C_2g(t)}\right)\right|^2 = B|\cosh(\alpha+i\beta)|^2,$$

where α, β are defined in (1.38) and

$$B = 4|C_1C_2|e^{\int_0^t \Re(\theta_1+\theta_2)dy} = \frac{4|a_{11}(0)-\theta_1(0)||a_{11}(0)-\theta_2(0)|}{|\theta_1(0)-\theta_2(0)|^2}e^{\int_0^t \Re(\theta_1+\theta_2)dy}.$$

Formula (1.40) follows from this formula in view of

$$|\cosh(\alpha+i\beta)|^2 = \sinh^2(\alpha)+\cos^2(\beta).$$

□

Proof of (1.44). By direct calculations

$$g(t) = g(0)-2\int_0^t a_{12}(t)e^{\int_0^t 2\varphi(y)dy} = g(0)-2\int_0^t \varphi e^{\int_0^s 2\varphi(z)dz} = -e^{\int_0^s 2\varphi(z)dz},$$

$$\xi = -\frac{g'(t)}{2g(t)} = -\varphi(t), \quad \theta_1 = \xi+\frac{a'_{12}}{2a_{12}}-\frac{\xi'}{2\xi} = -\varphi+\frac{\varphi'}{2\varphi}-\frac{\varphi'}{2\varphi} = -\varphi, \quad \theta_2 = \varphi,$$

$$B(t) = \frac{4|\theta_1(0)\theta_2(0)|}{|\theta_1(0)-\theta_2(0)|^2}e^{\int_0^t \Re(\theta_1+\theta_2)dy} = 1, \quad \varphi = iW\cos(t\omega)[\cos(tE)+i\sin(tE)],$$

$$\alpha(t) = \frac{1}{2}\Re\ln\left(\frac{-g(0)\theta_2(0)}{g(t)\theta_1(0)}\right) = -\frac{1}{2}\Re\ln\left(\frac{g(t)}{g(0)}\right) = -\frac{1}{2}\Re\ln\left(e^{i\pi+\int_0^t 2\varphi}\right),$$

$$\alpha(t) = -\frac{1}{2}\Re(i\pi+\int_0^t 2\varphi ds) = -\int_0^t \Re[\varphi]ds = W\int_0^t \cos(s\omega)\sin(sE)ds,$$

$$\beta(t) = -\frac{1}{2}\Im(i\pi+\int_0^t 2\varphi ds) = -\frac{\pi}{2}-W\int_0^t \cos(s\omega)\cos(sE)ds.$$

□

Proof of (1.50). From (1.37) we have

$$g_0(t) = g(0) - 2iW\int_0^t e^{is(\omega+\Delta)}\cos(s\omega)ds = \frac{-2W[(\Delta+\omega)\cos(t\omega) - i\omega\sin(t\omega)]e^{it(\Delta+\omega)}}{\Delta(\Delta+2\omega)}$$

$$\xi = -\frac{g'(t)}{2g(t)} = -\frac{i\Delta(\Delta+2\omega)\cos(t\omega)}{2(\Delta+\omega)\cos(t\omega) - 2i\omega\sin(t\omega)}, \quad m = \frac{2\Delta(\Delta+2\omega)}{(\Delta+\omega)(\Delta+\omega-E)} \geq 1,$$

$$\theta_1 = \xi + \frac{a'_{12}}{a_{12}} - \frac{\xi'}{2\xi} = \frac{i(E-\Delta-\omega)}{2}, \quad \theta_2 = \frac{i}{2}\left(E - \Delta - \omega + \frac{2\Delta(\Delta+2\omega)\cos(t\omega)}{(\Delta+\omega)\cos(t\omega) - i\omega\sin(t\omega)}\right)$$

$$B = B(t) = \frac{4|\theta_1(0)\theta_2(0)|}{|\theta_1(0)-\theta_2(0)|^2}e^{\int_0^t \Re(\theta_1+\theta_2)dy} = \frac{(\Delta+\omega-E)^2(m-1)(\Delta+\omega)\sqrt{(\Delta+\omega)^2+\omega^2}}{\Delta^2(\Delta+2\omega)^2} \tag{2.32}$$

$$\alpha = \frac{1}{2}\Re\ln\left(\frac{e^{it(\Delta+\omega)}[i\omega\sin(t\omega) - (\Delta+\omega)\cos(t\omega)]}{(\Delta+\omega)(m-1)}\right) = \frac{1}{2}\ln\frac{R}{(\Delta+\omega)(m-1)} \tag{2.33}$$

$$\beta = \frac{1}{2}\Im\ln\left(\frac{e^{it(\Delta+\omega)}[i\omega\sin(t\omega) - (\Delta+\omega)\cos(t\omega)]}{(\Delta+\omega)(m-1)}\right) = \frac{t(\Delta+\omega) - \eta(t)}{2} \tag{2.34}$$

where

$$i\omega\sin(t\omega) - (\Delta+\omega)\cos(t\omega) = Re^{-i\eta(t)}.$$

□

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