POISSON’S INEQUALITY FOR A DIRICHLET PROBLEM ON A TIME SCALE

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This paper is dedicated to Professor Lynn Erbe

ABSTRACT. By separation of variables we derive Poisson’s inequality for a Dirichlet problem in a circle on a time scale.

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1. Introduction. Poisson’s inequality

The fundamental Poisson’s formula characterizes solutions of the Dirichlet problem by their averages over circles (see [7]). Many generalizations of this formula have been proven (see for example [2, 3, 5, 6]), including extensions to partial differential equations on manifolds. In this paper we prove the version of Poisson’s formula on a time scale, introduced by Hilger in [4].

Consider the Dirichlet problem in a circle with radius \( r_0 > 0 \)

\[
\begin{align*}
    u^\Delta^\Delta(r, \varphi) + \frac{u^\Delta(r, \varphi)}{\sigma(r)} + \frac{u_{\varphi\varphi}(r, \varphi)}{r\sigma(r)} &= 0, \quad 0 < r < r_0, \quad 0 \leq \varphi \leq 2\pi \\
    u(r_0, \varphi) &= f(\varphi), \quad 0 \leq \varphi \leq 2\pi.
\end{align*}
\] (1.1) (1.2)

Here \( u^\Delta(r, \cdot) \) is the delta derivative (see [4, 1]) with respect to the variable \( r \) on a time scale \( \mathbb{T} \). Also, \( u_{\varphi}(\cdot, \varphi) \) denotes the partial derivative of the function \( u(r, \varphi) : \mathbb{T} \times [0, 2\pi] \to \mathbb{R} \). The forward jump operator is defined by \( \sigma(r) = \inf\{s \in \mathbb{T}, s > r\} \), where \( r \in \mathbb{T} \).

The set of functions \( p(r, \varphi) : \mathbb{T} \times [0, 2\pi] \to \mathbb{R} \) that are rd-continuous in \( r \in \mathbb{T} \) and continuous in the variable \( \varphi \in [0, 2\pi] \) will be denoted by \( C_{rd} \). The set of functions \( p(r, \varphi) \) such that their n-th delta derivative with respect to the variable \( r \) exists and is rd-continuous for \( r \in \mathbb{T} \), and their m-th derivative with respect to \( \varphi \in [0, 2\pi] \) exists and is continuous on \([0, 2\pi]\) is denoted by \( C_{rd}^{(n,m)} \). We say that the real-valued function

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$p(r, \varphi)$ is regressive on $\mathbb{T} \times [0, 2\pi]$ if $1 + \mu(r)p(r, \varphi) \neq 0$ for all $r \in \mathbb{T}, \varphi \in [0, 2\pi]$. The set of regressive functions on $\mathbb{T} \times [0, 2\pi]$ that belong to $C_{rd}$ is denoted by $\mathcal{R}$.

If $1 + \mu p \neq 0$, then the (generalized) exponential function $e_p(t, t_0)$ is the unique solution of the initial value problem

$$x^\Delta = p(t)x(t), \quad x(t_0) = 1,$$

and is given by the formula (see [4, 1])

$$e_p(t, t_0) = \exp \left[ \int_{t_0}^{t} \lim_{q \to \mu(s)} \frac{\log(1 + qp(s))}{q} \Delta s \right] \quad (1.3)$$

where $\log$ is the principal logarithmic function.

The set $\mathcal{R}$ along with the addition $\oplus$ defined by

$$p \oplus q := p + q + \mu p q \quad (1.4)$$

forms an Abelian group called the regressive group (see [1].) By $\mathcal{R}^{(n,m)}$ we denote the set of regressive functions that belong to $C^{(n,m)}_{rd}$. Note that (see [1])

$$e_p(r, r_0)e_q(r, r_0) = e_{p\oplus q}(r, r_0), \quad e_p(r, r_0)e_q(r, r_0) = e_{p\oplus q}(r, r_0), \quad p \oplus q = \frac{p - q}{1 + \mu q}. \quad (1.5)$$

Define the auxiliary functions

$$A(r) = \exp \int_{r_0}^{r} \left[ \lim_{q \to \mu(y)} \left( \frac{\ln(q/y)}{q} \right) \right] \Delta y, \quad B(r) = \sum_{n=2}^{\infty} \frac{1}{n^2 \lim_{q \to \mu(y)} \left( \frac{1}{q} \right) \Delta y}, \quad (1.6)$$

and

$$K(r) = \exp \int_{r}^{r_0} \left[ \lim_{q \to \mu(y)} \left( \frac{\ln(1 + q/y)}{q} \right) \right] \Delta y. \quad (1.7)$$

Note that $B(r)$ is the analogue of the shifted Riemann zeta function over the real.

**Theorem 1.1.** Assume

$$\int_{r}^{r_0} \lim_{q \to \mu(y)} \left( \frac{1}{q} \right) \Delta y = \int_{r}^{r_0} \frac{\Delta y}{\mu(y)} > 1, \quad r < r_0. \quad (1.8)$$

Then the solutions $u \in C^{(2,2)}_{rd}$ of the Dirichlet problems (1.1), (1.2) satisfy Poisson’s inequality

$$\left| u(r, \varphi) - \int_{0}^{2\pi} P_1(r, \alpha)f(\alpha)d\alpha \right| \leq \frac{A(r)B(r)}{\pi} \int_{0}^{2\pi} |f(\alpha)|d\alpha, \quad (1.9)$$

where

$$P_1(r, \alpha) = \frac{K^2(r) - 1}{2\pi [(K(r) - 1)^2 + 4K(r)\sin^2((\alpha - \varphi)/2)]} \quad (1.10)$$

is the Poisson kernel.

**Remark 1.2.** For the continuous time scale we have $\mu(r) \to 0$, $A(r) \to 0$, $K(r) \to r_0/r$ and hence we get Poisson’s formula

$$u(r, \varphi) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(r_0^2 - r^2)f(\alpha)d\alpha}{r_0^2 - 2r_0r \cos(\alpha - \varphi) + r^2}. \quad (1.11)$$
Example 1.3. Consider the discrete time scale
\[ T_1 = \left\{ 0, \frac{r_0}{n}, \frac{2r_0}{n}, \ldots, \frac{r_0 n}{n} \right\} = \{ t_k \}_{k=0}^{n}, \quad y = t_k = \frac{kr_0}{n} = k\mu, \quad k = 0, \ldots, n. \] (1.12)
Then
\[ \sigma(t) = t + \frac{r_0}{n}, \quad \mu(t) = \frac{r_0}{n}, \quad r = \frac{m r_0}{n}, \quad m \geq 1. \] (1.13)
By taking \( f(y) = \frac{1}{n} \ln(1 + \mu/y) \) we get
\[ K(r) = \exp \int_r^{r_0} f(y) \Delta y = \exp \sum_{k=m}^{n-1} \mu(t) f(t_k) = \prod_{k=m}^{n-1} \left( 1 + \frac{1}{k} \right), \] (1.14)
\[ A(r) = \exp \sum_{k=m}^{n-1} \ln(1/k) = \prod_{k=m}^{n-1} \frac{1}{k}, \quad B(r) = \sum_{n=2}^{\infty} n^{m-n}. \] (1.15)
Assumption (1.8) is equivalent to \( n - m = n - \frac{nr_0}{r_0} > 1 \) or \( r_0 > r \). Thus (1.9) gives
\[ \left| u(r, \varphi) - \int_0^{2\pi} P_1(r, \alpha) f(\alpha) d\alpha \right| \leq \frac{A(r)B(r)}{\pi} \int_0^{2\pi} |f(\alpha)| d\alpha. \] (1.16)
Or for \( m < n - 1 \),
\[ \left| u(r, \varphi) - \int_0^{2\pi} P_1(r, \alpha) f(\alpha) d\alpha \right| \leq \frac{1}{\pi} \sum_{k=2}^{\infty} n^{m-n} \left( \prod_{k=m}^{n-1} \frac{1}{k} \right) \int_0^{2\pi} |f(\alpha)| d\alpha, \] (1.17)
where the Poisson kernel \( P_1 \) is given by
\[ P_1 = \frac{1}{2\pi} \frac{\prod_{k=m}^{n-1} (1 + \frac{1}{k})^{-2} - 1}{(\prod_{k=m}^{n-1} (1 + \frac{1}{k}) - 1)^2 + 4 \prod_{k=m}^{n-1} (1 + \frac{1}{k}) \sin^2((\alpha - \varphi)/2)}. \] (1.18)

2. Proof

By seeking a solution of (1.1) in the form of \( u = R(r) \Phi(\varphi) \) we get from (1.1)
\[ R^{\Delta \Delta}(r) \Phi(\varphi) + \frac{R^{\Delta}(r) \Phi(\varphi)}{\sigma(r)} + \frac{R(r) \Phi''(\varphi)}{r \sigma(r)} = 0. \] (2.1)
Separating the variables leads to
\[ \frac{r \sigma(r) R^{\Delta \Delta}(r)}{R(r)} + \frac{r R^{\Delta}(r)}{R(r)} = - \frac{\Phi''(\varphi)}{\Phi(\varphi)} = n^2 = \text{const}, \] (2.2)
or
\[ r \sigma(r) \frac{R^{\Delta \Delta}(r)}{R(r)} + \frac{R^{\Delta}(r)}{R(r)} - n^2 = 0, \] (2.3)
and
\[ \Phi''(\varphi) = -n^2 \Phi(\varphi). \] (2.4)
Solutions of (2.4) are given by
\[ \Phi_n(\varphi) = a_n \sin(n \varphi) + b_n \cos(n \varphi), \quad n = 0, 1, 2, \ldots. \]
We seek solutions of (2.3) in the form of the exponential function on a time scale $R(r) = e_{\lambda/r}(r, r_0)$ (see (1.3)). A substitution of $R(r)$ in (2.3) gives

$$\lambda^2 - \lambda + \lambda - n^2 = 0, \quad \Rightarrow \lambda_1 = -n, \quad \lambda_2 = n. \quad (2.5)$$

Ignoring solutions corresponding to $\lambda_2 = -n$, (since $e_{-n/r}(r)$ could be unbounded as $r \to 0$) we get $R(r) = R_n(r) = e_n/r(r, r_0)$, and

$$u(r, \varphi) = \sum_{n=0}^{\infty} R_n(r) \Phi_n(\varphi) = \sum_{n=0}^{\infty} [a_n \sin(n\varphi) + b_n \cos(n\varphi)] e_{n/r}(r, r_0). \quad (2.6)$$

From the boundary condition (1.2) we have

$$f(\varphi) = \sum_{n=0}^{\infty} [a_n \sin(n\varphi) + b_n \cos(n\varphi)] = b_0 + \sum_{n=1}^{\infty} [a_n \sin(n\varphi) + b_n \cos(n\varphi)]. \quad (2.7)$$

Multiplying (2.7) by $\sin(m\varphi), \cos(m\varphi), \quad m = 0, \pm 1, \pm 2, \cdots$ and then integrating each corresponding expression yields to

$$\int_0^{2\pi} f(\varphi) \sin(m\varphi) d\varphi = a_m \int_0^{2\pi} \sin^2(m\varphi) d\varphi = a_m \pi, \quad \int_0^{2\pi} f(\varphi) \cos(m\varphi) d\varphi = b_m \int_0^{2\pi} \cos^2(m\varphi) d\varphi = b_m \pi,$$

and

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \sin(m\alpha) d\alpha, \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \cos(m\alpha) d\alpha, \quad m = 1, 2, \ldots,$$

$$b_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) d\alpha.$$

Substitution of these formulas in (2.6) gives

$$u(r, \varphi) = \sum_{n=1}^{\infty} \left( \frac{e_{n/r}(r, r_0)}{\pi} \int_0^{2\pi} f(\alpha) [\sin(n\alpha) \sin(n\varphi) + \cos(n\alpha) \cos(n\varphi)] d\alpha \right)$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) d\alpha$$

$$u(r, \varphi) = \sum_{n=1}^{\infty} \frac{e_{n/r}(r, r_0)}{\pi} \int_0^{2\pi} f(\alpha) \cos(n\alpha - n\varphi) d\alpha + \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) d\alpha$$

or

$$u(r, \varphi) = \int_0^{2\pi} f(\alpha) P(r, \varphi, \alpha) d\alpha, \quad (2.8)$$

where

$$P(r, \varphi, \alpha) = \frac{1}{\pi} \left( \frac{1}{2} + \sum_{n=1}^{\infty} e_{n/r}(r, r_0) \cos(n\alpha - n\varphi) \right). \quad (2.9)$$

Define the Poisson kernel by the formula

$$P_1(r, \varphi, \alpha) = \frac{1}{\pi} \left( \frac{1}{2} + \sum_{n=1}^{\infty} e_{1/r}(r, r_0) \cos(n\alpha - n\varphi) \right). \quad (2.10)$$
Moreover, by assuming condition (1.8) we get

\[ P(r, \varphi, \alpha) = P_1(r, \varphi, \alpha) + Q_2(r, \varphi, \alpha), \]  

(2.11)

where

\[ Q_2(r, \varphi, \alpha) = \sum_{n=2}^{\infty} S_n(r) \cos(n \alpha - n \varphi), \quad S_n(r) = \frac{e^{n/r}(r, r_0) - e^{n/r}_1(r, r_0)}{\pi}. \]  

(2.12)

Thus from (2.8) we get

\[ u(r, \varphi) - \int_0^{2\pi} P_1(r, \varphi, \alpha) f(\alpha) d\alpha = \int_0^{2\pi} Q_2(r, \varphi, \alpha) f(\alpha) d\alpha. \]  

(2.13)

Further since \( e^{1/r}(r, r_0) = \frac{1}{K(r)} < 1 \) from (2.10) we get

\[ P_1(r, \varphi, \alpha) = \frac{1}{\pi} \left( \frac{1}{2} + \Re \left[ \sum_{n=1}^{\infty} \left( \frac{e^{i(\alpha-\varphi)}}{K(r)} \right)^n \right] \right). \]  

(2.14)

By the geometric progression sum formula we get

\[ 2\pi P_1(r, \varphi, \alpha) = 2\Re \left[ \frac{1}{1 - e^{i(\alpha-\varphi)} K(r)} \right] - 1, \]

\[ 2\pi P_1(r, \varphi, \alpha) = \frac{K(r)}{K(r) - e^{i(\alpha-\varphi)}} + \frac{K(r)}{K(r) - e^{-i(\alpha-\varphi)}} - 1 \]

\[ = \frac{K^2(r) - 1}{1 - 2K(r) \cos(\alpha - \varphi) + K^2(r)}, \]

and

\[ P_1(r, \varphi, \alpha) = \frac{K^2(r) - 1}{2\pi(K^2(r) - 2K(r) \cos(\alpha - \varphi) + 1)} \]  

(2.15)

or (1.10).

**Lemma 2.1.**

\[ e^{n/r}_1(r, r_0) = e^{n/r}_p(r, r_0), \quad p_n = \frac{(1 + \mu/r)^n - 1}{\mu}, \quad n = 1, 2, \ldots, \]  

(2.16)

\[ e^{n/r}_1(r, r_0) \leq e^{n/r}_n(r, r_0), \quad n = 1, 2, \ldots, \quad r \leq r_0, \]  

(2.17)

\[ e^{n/r}_n(r, r_0) \leq \begin{cases} \frac{A(r)n^{-j_{\nu_0}(\Delta \nu)}}{\mu(y)}, & \mu(y) > 0, \\ \left( \frac{1}{r_0} \right)^n, & \mu(y) \equiv 0, \end{cases} \quad n = 2, 3, \ldots, \quad r \leq r_0, \]  

(2.18)

\[ \pi |S_n| = \pi S_n \leq \begin{cases} A(r)n^{-j_{\nu_0}(\Delta \nu)} - K^{-n}, & \mu(y) > 0, \\ 0, & \mu(y) \equiv 0. \end{cases} \]  

(2.19)

Moreover, by assuming condition (1.8) we get

\[ |\pi Q_2| \leq \begin{cases} A(r)B(r) - \frac{1}{K(r)(K(r)-1)}, & \mu(y) > 0, \\ 0, & \mu(y) \equiv 0. \end{cases} \]  

(2.20)
**Proof of Lemma 2.1:** In view of (1.3), expression (2.16) is true when \( n = 1 \). On the other hand, if we assume (2.16), then in view of (1.5) we have

\[
e_{1/\mathbf{r}}(r, r_0) = e_1(r, r_0)e_p(r, r_0) = e_q(r, r_0),
\]

where

\[
q = \frac{1}{\mathbf{r}} + \frac{1}{\mu}[(1 + \mu/\mathbf{r})^n - 1] + \frac{\mu}{\mathbf{r}\mu}[(1 + \mu/\mathbf{r})^n - 1]
\]

\[
= \frac{1}{\mu}(1 + \mu/\mathbf{r})^n + \frac{1}{\mathbf{r}}(1 + \mu/\mathbf{r})^n - \frac{1}{\mu}
\]

\[
= \frac{1}{\mu}[(1 + \mu/\mathbf{r})^{n+1} - 1] = p_{n+1}.
\]

The rest of the proof follows by induction.

The inequality (2.17) follows from (2.16) and hence \( p_n \geq n/\mathbf{r} \). To prove the inequality (2.18) we note that if \( \mu(y) > 0 \) then (1.3) implies that

\[
e_{n/r}(r, r_0) = \exp \int_{r_0}^{r} \frac{\ln(n) + \ln(1/n + \mu/r)}{\mu} \Delta r
\]

\[
= n^{\int_{r_0}^{r} \Delta r/\mu} \exp \int_{r_0}^{r} \frac{\ln(1/n + \mu/r)}{\mu} \Delta r
\]

\[
\leq n^{\int_{r_0}^{r} \Delta y/\mu(y)} \exp \int_{r_0}^{r} \frac{\ln(\mu/r)}{\mu} \Delta r
\]

\[
= A(r)n^{\int_{r_0}^{r} \Delta y/\mu(y)}.
\]

For the case \( \mu(y) \equiv 0 \), inequalities (2.18), (2.19) are obvious. If \( \mu(y) > 0 \) we obtain (2.19) by using (2.18). To see this, note that

\[
\pi|S_n| = \pi S_n = e_{n/r}(r, r_0) - e_{1/r}(r, r_0) = e_{n/r}(r, r_0) - K^{-n} \leq A(r)n^{\int_{r_0}^{r} \Delta y/\mu(y)} - K^{-n}.
\]

Using the assumption (1.8) we get (2.20). That is

\[
|\pi Q_2| \leq \sum_{n=2}^{\infty} \pi|S_n| \leq A(r) \sum_{n=2}^{\infty} n^{\int_{r_0}^{r} \Delta y/\mu(y)} - \sum_{n=2}^{\infty} K^{-n} = A(r)B(r) - \frac{1}{K(r)(K(r) - 1)}.
\]

Finally, from (2.13),(2.19) we get

\[
\pi \left| u(r, \varphi) - \int_{0}^{2\pi} P_1(r, \varphi, \alpha)f(\alpha)d\alpha \right| \leq \int_{0}^{2\pi} \pi|Q_2(r, \varphi, \alpha)f(\alpha)|d\alpha \leq \left( A(r)B(r) - \frac{1}{K(r)(K(r) - 1)} \right) \int_{0}^{2\pi} |f(\alpha)|d\alpha \leq A(r)B(r) \int_{0}^{2\pi} |f(\alpha)|d\alpha,
\]

which gives (1.9).

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