ON A POSTERIORI ERROR ESTIMATES FOR ONE-DIMENSIONAL CONVECTION-DIFFUSION PROBLEMS

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January 24, 2005

Abstract

This paper is concerned with the upwind finite-difference discretization of a quasilinear singularly perturbed boundary value problem without turning points. Kopteva's a posteriori error estimate [N. Kopteva, Maximum norm a posteriori error estimates for a one-dimensional convection-diffusion problem, *SIAM J. Numer. Anal.*, **39**, 423–441 (2001)] is generalized and improved.

2000 MSC: 65L10, 65L70.
Keywords: Convection-diffusion, quasilinear boundary value problem, singular perturbation, finite difference scheme, Bakhvalov mesh, Shishkin mesh, a posteriori error estimate.

1 INTRODUCTION

We consider the singularly perturbed quasilinear convection-diffusion problem

$$Tu := -\varepsilon u'' - b(x,u)' + c(x,u) = 0 \quad \text{for} \quad x \in X := [0,1], \quad u(0) = u(1) = 0, \tag{1}$$

where ε is the perturbation parameter, $0 < \varepsilon \ll 1$, and b and c are two $C^2(X \times I\!R)$ functions satisfying

$$b_u(x,u) \geq \beta > 0, \quad c_u(x,u) \geq \gamma, \quad x \in X, \quad u \in I\!R, \tag{2}$$

where $\gamma \leq 0$. Since we assume that ε is small enough, it follows that $\beta^2 + 4\varepsilon\gamma > 0$ and then by [1] the problem (1) has a unique solution $u_\varepsilon \in C^3(X)$. This solution in general exhibits a boundary layer of exponential type near $x = 0$ and its derivatives can be estimated as in [2],

$$|u_\varepsilon^{(k)}(x)| \leq M\left(1 + \varepsilon^{-k} e^{-\beta x/\varepsilon}\right), \quad x \in X, \quad k = 0,1,2. \tag{3}$$

Here an throughout the paper, M denotes any (in the sense of $O(1)$) positive constant which is independent of ε and of the number of mesh points used when (1) is solved numerically. Thus, M may have different values in different inequalities.

It moreover holds (cf. [3]) that

$$|u_\varepsilon(x) - u_0(x)| \leq M\left(\varepsilon + e^{-\beta x/\varepsilon}\right), \quad x \in X, \tag{4}$$

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where u_0 is the unique $C^2(X)$ solution of the reduced problem

$$-b(x,u)' + c(x,u) = 0, \quad x \in X, \quad u(1) = 0. \tag{5}$$

Singularly perturbed boundary-value problems arise in many applications, see [4] and [5] for instance. The problem (1) has been used frequently as a model for testing different numerical methods for singular perturbation problems. In addition to the above mentioned papers [2] and [3], some other more recent papers dealing with the numerical solution of (1) are [6], [7], and [8]. We are interested here in one of Kopteva's results in [7], where the special case in which $c_u \equiv 0$ is considered. We represent this case by writing also $c(x,u) = -f(x)$. Kopteva's result is an a posteriori error estimate for the numerical solution of (1) with $c(x,u) = -f(x)$, obtained by the first-order upwind scheme. The error estimate is derived under the less restrictive smoothness assumptions $b \in C^1(X \times I\!R)$ and $f \in C^1(X)$.

In section 2, after introducing some further notation, we show that Kopteva's approach can be applied to the general case $c_u \not\equiv 0$ as well. However, we complete the derivation of the a posteriori error estimate differently, viz. we expand it and ignore all the terms of order higher than one. We do this first in section 3 for the special case $c(x,u) = -f(x)$ and then in section 4 for the general case. In both cases, we need smoother functions b and c (as indicated in our assumptions above) and we make use of the special discretization meshes of Bakhvalov or Shishkin types. The general case requires also that the reduced solution u_0 be taken into account. Finally, in section 5 we present results of some numerical experiments.

2 PRELIMINARIES

Let X^N be a general discretization mesh with points x_i, $i = 0, 1, \dots, N$, where $0 = x_0 < x_1 < \dots < x_N = 1$. Let also $X_i = [x_{i-1}, x_i]$, $i = 1, 2, \dots, N$, $h_i = x_i - x_{i-1}$, $i = 1, 2, \dots, N$, $\hbar_i = (h_i + h_{i+1})/2$, $i = 1, 2, \dots, N-1$, $\hbar_N = h_N/2$, and $h = \max_i h_i$.

We consider two special discretization meshes, both dense in the boundary layer. The first one belongs to the meshes of Bakhvalov type. It is generated by a suitable function λ so that $x_i = \lambda(i/N)$, $i = 0, 1, \dots, N$. A general description of mesh generating functions can be found in [9] or [6] for instance. For simplicity, we consider here specifically

$$\lambda(t) = \begin{cases} \varphi(t) := \varepsilon t/(q-t) & \text{if } 0 \le t \le \alpha, \\ \psi(t) := \varphi'(\alpha)(t-\alpha) + \varphi(\alpha) & \text{if } \alpha \le t \le 1, \end{cases}$$

cf. [9] and [3]. q is here a mesh parameter, a fixed number in the interval $(\varepsilon, 1)$, and α is the unique number guaranteeing that $\psi(1) = 1$. Thus, λ is a strictly increasing $C^1(X)$ function which maps X onto itself. Let X_B^N denote the discretization mesh generated by the specified λ.

The other mesh is of Shishkin type. Shishkin meshes are piecewise equidistant and therefore simpler, see [6] or [3] for instance. However, they produce somewhat less accurate results than Bakhvalov meshes, cf. (7)-(8) below. For the problem (1), a Shishkin mesh consists of two equidistant parts, one fine over the interval $[0, \tau]$, and the other coarse over $[\tau, 1]$. τ is here the transition point between the fine and the coarse parts of the mesh, $\tau = (2\varepsilon/\beta)\ln N$. Then,

$$x_i = \begin{cases} 2\tau i/N & \text{for } i = 0, 1, \dots, N/2, \\ \tau + (1-\tau)(2i/N - 1) & \text{for } i = N/2, N/2+1, \dots, N, \end{cases}$$

where we assume for simplicity that N is even. Let this mesh be denoted by X_S^N.

For both types of meshes, $h = h_N \leq M/N$.

By $w^N = \{w_i^N\}$ we denote an arbitrary mesh function on X^N. For any mesh function we assume that $w_0^N = w_N^N = 0$. We discretize the problem (1) using the standard upwind scheme, also known as the Engquist-Osher scheme [10],

$$T^N w_i^N := -\frac{\varepsilon}{\hbar_i}(D_+ w_i^N - D_- w_i^N) - \frac{\delta_+ b(x_i, w_i^N)}{\hbar_i} + c(x_i, w_i^N) = 0, \quad i = 1, 2, \dots N-1, \tag{6}$$

where

$$D_+ w_i^N = \frac{\delta_+ w_i^N}{h_{i+1}}, \quad D_- w_i^N = \frac{\delta_- w_i^N}{h_i},$$

and

$$\delta_+ w_i^N = w_{i+1}^N - w_i^N, \quad \delta_- w_i^N = w_i^N - w_{i-1}^N.$$

By [6], the discrete problem (6) has a unique solution, which we denote by $w_\varepsilon^N = \{w_{\varepsilon,i}^N\}$. This solution is bounded uniformly with respect to ε. Let u^N denote the piecewise linear interpolant of w_ε^N. Thus, $u^N \in C(X)$, it is a linear function on each interval X_i and $u^N(x_i) = w_{\varepsilon,i}^N$ for $i = 0, 1, \dots, N$. If the special meshes are used, the following holds true according to [8] (the same is proved in [6] but that proof requires a smoother function b):

$$|w_{\varepsilon,i}^N - u_\varepsilon(x_i)| \leq M\frac{L}{N}, \quad i = 0, 1, \dots, N, \tag{7}$$

where

$$L = \begin{cases} 1 & \text{if } X^N = X_B^N \\ \ln N & \text{if } X^N = X_S^N \end{cases}. \tag{8}$$

Another property of the special meshes is

$$|u_\varepsilon(x) - u_\varepsilon(x_{i-1})| \leq M\frac{L}{N}, \quad x \in X_i. \tag{9}$$

Analogously to the following form of the differential equation in (1):

$$-(Au)' = 0, \quad Au = \varepsilon u' + b(x, u) + \int_x^1 c(t, u(t))\, dt,$$

the discretization (6) can be written down as

$$T^N w_i = -\frac{A^N w_{i+1}^N - A^N w_i^N}{\hbar_i} = 0, \quad i = 1, 2, \dots, N-1, \tag{10}$$

with

$$A^N w_i^N = \varepsilon D_- w_i^N + b(x_i, w_i^N) + \sum_{j=i}^{N} \hbar_i c(x_j, w_j^N), \quad i = 1, 2, \dots, N.$$

This form of the scheme is similar to the one in [8], which uses a more general definition of $\hbar_i$. In [7], on the other hand, the operator A^N is defined in a slightly more general way (for the case $c(x, u) = -f(x)$ considered there). However, neither generalization is essential and we do not consider them here.

Kopteva [7] considers the following special case of (1),

$$\tilde{T}u := -\varepsilon u'' - b(x, u)' = f(x), \quad x \in X, \quad u(0) = u(1) = 0. \tag{11}$$

The following result is crucial in her error analysis:

$$\|u^N - u_\varepsilon\|_\infty \le \frac{2}{\beta}\|\tilde{T}u^N - f\|_*, \tag{12}$$

where

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in X} |u(x)|, \quad \|u\|_* = \min_{U:U'=u} \|U\|_\infty.$$

The estimate (12) immediately gives

$$\|u^N - u_\varepsilon\|_\infty \le \frac{2}{\beta}\|\eta'\|_* \tag{13}$$

for the general problem, where for $x \in X_i$, $i = 1, 2, \ldots, N$,

$$\eta(x) = -\varepsilon(u^N)'(x) - b(x, u^N(x)) + C - \int_x^1 c(t, u_\varepsilon(t))\, dt$$

with an arbitrary constant C. Thus,

$$\eta'(x) = Tu^N(x) - c(x, u^N(x)) + c(x, u_\varepsilon(x)).$$

Since from (13) it follows that

$$\|u^N - u_\varepsilon\|_\infty \le \frac{2}{\beta}\|\eta\|_\infty, \tag{14}$$

the a posteriori error estimate depends on how η is estimated.

We now transform $\eta(x)$ for $x \in X_i$ analogously to [7]. First we choose C as $C = A^N u^N(x_N)$ so that, according to (10), $A^N u^N(x_i) = C$ for all $i = 1, 2, \ldots, N-1$. Then we use the fact that $(u^N)'(x) = D_- u^N(x_i)$ for $x \in X_i$. We get

$$\begin{aligned} \eta(x) &= -\varepsilon D_- u^N(x_i) - b(x, u^N(x)) + A^N u^N(x_i) - \int_x^1 c(t, u_\varepsilon(t))\, dt \\ &= b(x, u^N(x_i)) - b(x, u^N(x)) + \sum_{j=i}^{N} \hbar_i c(x_j, u^N(x_j)) - \int_x^1 c(t, u_\varepsilon(t))\, dt. \end{aligned}$$

Therefore,

$$\eta(x) = \eta_1(x) + \eta_2(x),$$

where for $x \in X_i$

$$\eta_1(x) = \int_x^{x_i} b(t, u^N(t))'\, dt \tag{15}$$

and

$$\eta_2(x) = \sum_{j=i}^{N} \hbar_i c(x_j, u^N(x_j)) - \int_x^1 c(t, u_\varepsilon(t))\, dt. \tag{16}$$

In sections 3 and 4, we are going to use some approximate equalities ($\doteq$) and inequalities ($\dot{\le}$). They mean that the terms we omit are negligible relative to Mh.

3 THE CASE $c(x,u) = -f(x)$

In this section, we consider the problem (11). Kopteva's [7] error estimate is based on

$$\|\eta_2\|_\infty \le (\|f\|_\infty + \|f'\|_\infty)h.$$

and

$$\|\eta_1\|_\infty \le \bar{\beta} \max_{1\le i\le N} |\delta_- u^N(x_i)| + Bh,$$

where

$$\bar{\beta} \ge b_u(x,u), \quad x \in X, \quad u \in I\!R, \tag{17}$$

and

$$B = \max_{x\in X,\ |u|\le M_*} |b_x(x,u)|.$$

The above constant M_* results from an a priori estimate of the numerical solution,

$$\|u^N\|_\infty \le M_* := \frac{1}{\beta}\left[2\|b(\cdot,0)\|_\infty + \|f\|_\infty\right],$$

see [7]. The assumption (17) is not as serious a restriction as it may seem. It is introduced in this form for simplicity since it is possible to find an a priori domain containing u_ε and then the boundedness of b_u is guaranteed for u in that domain.

Thus, assuming that $b \in C^1(X \times I\!R)$ and $f \in C^1(X)$, and using (14), Kopteva proves the following a posteriori error estimate:

$$\|u^N - u_\varepsilon\|_\infty \le \frac{2}{\beta}\left[\bar{\beta} \max_{1\le i\le N} |\delta_- u^N(x_i)| + (B + \|f\|_\infty + \|f'\|_\infty)h\right]. \tag{18}$$

This estimate is valid on any mesh X^N.

We improve the estimate (18) by expanding and approximating both η_1 and η_2. We do this under the assumptions that the discretization mesh is either X_B^N or X_S^N and that b and f are smoother functions. Our approximation of η_1 is given in the following lemma.

Lemma 1. *Let $b \in C^2(X \times I\!R)$ and let the discretization mesh be either X_B^N or X_S^N. Then for $x \in X_i$, it holds that*

$$\eta_1(x) \doteq (x_i - x)\left[\frac{d}{dx} b(x, u^N(x))\right]_{x=x_i}.$$

Proof. Expand $b(t, u^N(t))'$ in (15) about x_i to get

$$\eta_1(x) = (x_i - x)\left[\frac{d}{dx} b(x, u^N(x))\right]_{x=x_i} + r_i,$$

where

$$|r_i| \le M h_i^2 \left[1 + |D_- u^N(x_i)| + (D_- u^N(x_i))^2\right].$$

The special mesh, (7), and (9) imply

$$|D_- u^N(x_i)| \le |D_-[u^N(x_i) - u_\varepsilon(x_i)]| + |D_- u_\varepsilon(x_i)| \le M\frac{L}{h_i N},$$

and therefore $|r_i| \le M(L/N)^2$ and this term can be ignored. ■

Lemma 2 deals with η_2. This result is true not only on the special meshes but on all meshes with $h \le M/N$.

Lemma 2. *Let $f \in C^2(X)$ and let the discretization mesh be either X_B^N or X_S^N. Then for $x \in X_i$, it holds that*

$$\eta_2(x) \doteq \left(x_i - x - \frac{h_i}{2}\right) f(x_i).$$

Proof. Upon replacing $c(x,u)$ in (16) with $-f(x)$, we get

$$\eta_2(x) = \int_x^1 f(t)\, dt - \sum_{j=i}^{N} \hbar_j f(x_j) = \zeta_1 + \zeta_2,$$

where

$$\zeta_1 = \int_x^{x_i} f(t)\, dt - \frac{h_i}{2} f(x_i)$$

and ζ_2 is the error of the trapezoidal formula for $\int_{x_i}^1 f(t)\, dt$. Therefore,

$$|\zeta_2| \le MNh^3 \le Mh^2.$$

Moreover, by expanding $f(t)$ in ζ_1 around x_i, it follows that

$$\eta_2 \doteq \zeta_1 \doteq (x_i - x) f(x_i) - \frac{h_i}{2} f(x_i).$$

■

We can now prove the main result of this section.

Theorem 1. *Let $b \in C^2(X \times I\!R)$, $f \in C^2(X)$, and let b satisfy the condition in (2). Let u_ε be the solution of (11) and let u^N be the linear interpolant of the numerical solution of (6) on X_B^N or X_S^N and with $c(x,u) = -f(x)$. Then the following approximate a posteriori error estimate holds true:*

$$\|u^N - u_\varepsilon\|_\infty \dot{\le} \frac{1}{\beta} \max_{1 \le i \le N} h_i \max\{|f(x_i)|, |2[b(x,u^N(x))']_{x=x_i} + f(x_i)|\}. \tag{19}$$

Proof. Combining the results of Lemmas 1 and 2, we get

$$\eta(x) \doteq (x_i - x)\left\{[b(x,u^N(x))']_{x=x_i} + f(x_i)\right\} - \frac{h_i}{2} f(x_i), \quad x \in X_i.$$

After maximizing the above right-hand side, we conclude that

$$\eta(x) \dot{\le} \max\left\{\frac{h_i}{2}|f(x_i)|, \left| h_i \left[b(x,u^N(x))'\right]_{x=x_i} + \frac{h_i}{2} f(x_i)\right|\right\}, \quad x \in X_i.$$

Then the assertion follows from (14). ■

Numerical results in section 5 confirm that the error estimate (19) is much sharper than Kopteva's (18). Another advantage of (19) is that it does not need the upper bounds for $|b_x|$, b_u, and $|f|$. Note that the values of $[b(x,u^N(x))']_{x=x_i}$ can be calculated easily after finding the numerical solution $\{w_{\varepsilon,i}^N\}$:

$$[b(x,u^N(x))']_{x=x_i} = b_x(x_i, u^N(x_i)) + b_u(x_i, u^N(x_i)) D_- u^N(x_i) = b_x(x_i, w_{\varepsilon,i}^N) + b_u(x_i, w_{\varepsilon,i}^N) D_- w_{\varepsilon,i}^N.$$

4 THE GENERAL CASE

We now return to the fully quasilinear problem (1). In this case, η_2 cannot be treated in the same way as in the previous section. Therefore, in this section we make use of u_0, the solution of the reduced problem, and assume that $\varepsilon \ll h$, which is not a serious practical restriction. Of course, the reduced solution may be used in other ways in numerical methods for the problem (1), see for instance [11] and [12]. We are interested here only in the numerical method given in (6) and in seeing how the error of its solution can be estimated using u_0. We assume that u_0 is known, but even if it is not, its numerical approximation of at least second order can be used instead.

We first replace u_ε in (16) with u_0. Because of (4), it follows that

$$\eta_2(x) \doteq \sum_{j=i}^{N} \hbar_j c(x_j, u^N(x_j)) - \int_x^1 c(t, u_0(t))\, dt, \quad x \in X_i.$$

Then the integral above can be modified and approximated like in the proof of Lemma 2. This gives

$$\eta_2(x) \doteq \sigma_i + \left(x - x_i + \frac{h_i}{2}\right) c(x_i, u_0(x_i)), \quad x \in X_i,$$

with

$$\sigma_i = \sum_{j=i}^{N} \hbar_j [c(x_j, u^N(x_j)) - c(x_j, u_0(x_j))].$$

Then we have the following generalization of Theorem 1.

Theorem 2. *Let $b, c \in C^2(X \times I\!R)$ and assume the condition (2). Let u_ε and u_0 be the solutions of (1) and (5) respectively. Also, let u^N be the linear interpolant of the numerical solution of (6) on X_B^N or X_S^N. Then, if $\varepsilon \ll 1/N$, the following approximate a posteriori error estimate holds true:*

$$\|u^N - u_\varepsilon\|_\infty \,\dot{\leq}\, \frac{1}{\beta} \max_{1\le i\le N} \max\{A_i, B_i\}, \tag{20}$$

where

$$A_i = |h_i c(x_i, u_0(x_i)) + 2\sigma_i|$$

and

$$B_i = |2h_i [b(x, u^N(x))']_{x=x_i} - h_i c(x_i, u_0(x_i)) + 2\sigma_i|.$$

The estimate (20) can be modified. For this, we need the following auxiliary result.

Lemma 3. *Let u_0 be the solution of the reduced problem (5) and let u^N be the linear interpolant of the numerical solution of (6) on X_B^N or X_S^N. Then,*

$$\left| \int_x^{x_i} [u^N(t)) - u_0(t)]\, dt \right| \le M\left(\frac{L}{N^2} + \varepsilon\right), \quad x \in X_i.$$

Proof. Let u_ε^N denote the linear interpolant of $\{u_\varepsilon(x_i)\}$. It follows that

$$\left| \int_x^{x_i} [u^N(t) - u_0(t)]\, dt \right| \le M(I_1 + I_2 + I_3),$$

where

$$I_1 = \left| \int_x^{x_i} [u^N(t) - u_\varepsilon^N(t)] \, dt \right|,$$

$$I_2 = \left| \int_x^{x_i} [u_\varepsilon^N(t) - u_\varepsilon(t)] \, dt \right|,$$

and

$$I_3 = \left| \int_x^{x_i} [u_\varepsilon(t) - u_0(t)] \, dt \right|.$$

It holds that $I_j \le ML/N^2$, $j = 1, 2$. For I_1, this follows from (7) and for I_2, from (9). Finally, $I_2 \le M(\varepsilon/N + \varepsilon)$ because of (4). ∎

Theorem 3. *Let $b, c \in C^2(X \times I\!R)$ and assume the condition (2). Let u_ε and u_0 be the solution of (1) and (5) respectively. Also, let u^N be the linear interpolant of the numerical solution of (6) on X_B^N or X_S^N. Then, if $\varepsilon \ll 1/N$, the following approximate a posteriori error estimate holds true:*

$$\|u^N - u_\varepsilon\|_\infty \dot{\le} \frac{1}{\beta} \max_{1 \le i \le N} \left[2\bar{\beta} |\delta_-[u^N(x_i) - u_0(x_i)]| + |h_i c(x_i, u_0(x_i)) + 2\sigma_i| \right]. \tag{21}$$

Proof. η_1, given in (15), can be rewritten as

$$\begin{aligned} \eta_1(x) &= \int_x^{x_i} [b(t, u^N(t))' \pm b(t, u_0(t))'] \, dt \\ &= \int_x^{x_i} [b(t, u^N(t)) - b(t, u_0(t))]' \, dt + \int_x^{x_i} c(t, u_0(t)) \, dt, \quad x \in X_i. \end{aligned}$$

Then it follows that

$$\eta(x) = \bar{\eta}_1(x) + \bar{\eta}_2(x),$$

with

$$\bar{\eta}_1(x) = \int_x^{x_i} [b(t, u^N(t)) - b(t, u_0(t))]' \, dt, \quad x \in X_i, \tag{22}$$

and

$$\bar{\eta}_2(x) = \eta_2(x) + \int_x^{x_i} c(t, u_0(t)) \, dt \doteq \sum_{j=i}^{N} \hbar_j c(x_j, u^N(x_j)) - \int_{x_i}^{1} c(t, u_0(t)) \, dt, \quad x \in X_i.$$

Using $Nh \le M$ and the trapezoidal formula again (cf. the proof of Lemma 2), we get the following approximation of $\bar{\eta}_2$:

$$\bar{\eta}_2(x) \doteq \sigma_i + \frac{h_i}{2} c(x_i, u_0(x_i)), \quad x \in X_i. \tag{23}$$

Let us now approximate $\bar{\eta}_1$. Because of Lemma 3, for $x \in X_i$, it follows from (22) that

$$\begin{aligned} \bar{\eta}_1(x) &\doteq \int_x^{x_i} [b_u(t, u^N(t))(u^N)'(t) - b_u(t, u_0(t)) u_0'(t)] \, dt \\ &\doteq \int_x^{x_i} b_u(t, u^N(t))[(u^N)'(t) - u_0'(t)] \, dt \\ &= \int_x^{x_i} b_u(t, u^N(t))[D_- u^N(x_i) - u_0'(t)] \, dt. \end{aligned}$$

We next replace $u_0'(t)$ with $D_-u_0(x_i)$ creating a negligible second-order error,

$$\bar{\eta}_1(x) \doteq [D_-(u^N(x_i) - u_0(x_i))] \int_x^{x_i} b_u(t, u^N(t))\, dt, \quad x \in X_i.$$

From this we get

$$|\bar{\eta}_1(x)| \mathrel{\dot{\le}} \bar{\beta}|\delta_-[u^N(x_i) - u_0(x_i)]|, \quad x \in X_i, \tag{24}$$

where $\bar{\beta}$ is given in (17). We complete the proof using (24) and (23). ■

5 NUMERICAL RESULTS

We consider three test problems, two linear ones and one nonlinear. The first linear problem is of the type described in (11),

$$-\varepsilon u'' - u' = f(x), \quad x \in X, \quad u(0) = u(1) = 0. \tag{25}$$

The second linear problem is with $c_u \not\equiv 0$,

$$-\varepsilon u'' - u' + u = g(x), \quad x \in X, \quad u(0) = u(1) = 0. \tag{26}$$

Both problems have the solution

$$u_\varepsilon(x) = \frac{(e-1)e^{-x/\varepsilon} - e + e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} + e^x,$$

which determines the functions f and g above. The nonlinear problem is a classical example due to O'Malley [13],

$$-\varepsilon u'' - e^u u' + \frac{\pi}{2} \sin\frac{\pi x}{2} e^{2u} = 0, \quad x \in X, \quad u(0) = u(1) = 0. \tag{27}$$

This problem has been used in many numerical experiments, including [7]. Its solution satisfies $u_\varepsilon(x) = u_A(x) + O(\varepsilon)$, where

$$u_A(x) = -\ln\left[\left(1 + \cos\frac{\pi x}{2}\right)\left(1 - \frac{1}{2}e^{-x/\varepsilon}\right)\right].$$

In all our numerical tests, we evaluate the exact maximum error,

$$E = E(N) = \max_{1 \le i \le N-1} |w_{\varepsilon,i}^N - \tilde{u}_\varepsilon(x_i)|,$$

where $\tilde{u}_\varepsilon = u_A$ for the nonlinear problem (27) and $\tilde{u}_\varepsilon = u_\varepsilon$ for the linear problems. We compare E to the a posteriori error estimates. If E^* denotes an a posteriori error estimate, then we calculate its efficiency as

$$\text{Eff} = E/E^*.$$

We expect that Eff ≤ 1. We also evaluate the numerical order of convergence,

$$\text{Ord} = \text{Ord}(N) = \log_2[E(N)/E(2N)].$$

We find Ord also for all a posteriori error estimates E^*.

We use the problem (25) to compare our estimate (19) to Kopteva's estimate (18). All the quantities needed for (18) are easy to find. The comparison is given in Tables 1–3 on different discretization meshes. It is clear that our estimate is superior to Kopteva's. In her paper [7], Kopteva does not calculate (18), but uses instead the quantity

$$\Delta = \max_{1 \le i \le N-1} |\delta_- w_{\varepsilon,i}^N|,$$

although there is no theoretical guarantee that Δ is an upper bound of the error. We too include Δ in all our tables. It is not surprising that Δ has the best efficiency, but we note that our theoretically safe estimate gets very close to Δ in some cases.

N	E	(19)	Eff	(18)	Eff	Δ	Eff
	Ord	Ord		Ord		Ord	
32	1.30E−1	2.85E−1	.46	9.99E−1	.13	1.60E−1	.81
	.94	.93		.99		.96	
64	6.75E−2	1.50E−1	.45	5.04E−1	.13	8.23E−2	.82
	.98	.96		.99		.98	
128	3.43E−2	7.70E−2	.45	2.54E−1	.14	4.18E−2	.82
	.99	.98		1.00		.99	
256	1.73E−2	3.90E−2	.44	1.27E−1	.14	2.11E−2	.82
	.99	.99		1.00		.99	
512	8.68E−3	1.96E−2	.44	6.36E−2	.14	1.06E−2	.82

Table 1. Errors, error estimates, their numerical orders of convergence, and error-estimate efficiency values for (25) solved on X_B^N with $q = 0.5$.

N	E	(19)	Eff	(18)	Eff	Δ	Eff
	Ord	Ord		Ord		Ord	
32	2.29E−1	4.25E−1	.54	2.43E+0	.09	3.63E−1	.63
	.97	1.00		.97		.89	
64	1.16E−1	2.12E−1	.55	1.24E+0	.09	1.96E−1	.59
	.99	1.00		.98		.94	
128	5.87E−2	1.06E−1	.55	6.29E−1	.09	1.02E−1	.57
	.99	1.00		.99		.97	
256	2.95E−2	5.31E−2	.56	3.16E−1	.09	5.21E−2	.57
	1.00	1.00		.99		.99	
512	1.48E−2	2.65E−2	.56	1.59E−1	.09	2.63E−2	.56

Table 2. Errors, error estimates, their numerical orders of convergence, and error-estimate efficiency values for (25) solved on X_B^N with $q = 0.8$.

N	E Ord	(19) Ord	Eff	(18) Ord	Eff	Δ Ord	Eff
32	1.71E−1	9.88E−1	.17	1.67E+0	.10	4.94E−1	.35
	.75	.51		.70		.51	
64	1.02E−1	6.92E−1	.15	1.03E+0	.10	3.46E−1	.29
	.79	.63		.74		.63	
128	5.89E−2	4.47E−1	.13	6.17E−1	.10	2.23E−1	.26
	.82	.72		.79		.71	
256	3.33E−2	2.72E−1	.12	3.57E−1	.09	1.36E−1	.24
	.85	.77		.82		.77	
512	1.85E−2	1.59E−1	.12	2.02E−1	.09	7.96E−2	.23

Table 3. Errors, error estimates, their numerical orders of convergence, and error-estimate efficiency values for (25) solved on X_S^N.

All the results shown here are obtained for $\varepsilon = 10^{-9}$. Due to the ε-uniformity of the numerical methods used, the results for other small values of ε are similar.

Comparing Tables 1–3, we can see that the Bakhvalov-type mesh X_B^N produces much better results than the Shishkin mesh X_S^N. On X_B^N, the error estimate (19) has greater efficiency for $q = 0.8$ than for $q = 0.5$. Greater values of the parameter q cause greater density of the mesh in the boundary layer. In the remaining tables, we use only X_B^N with $q = 0.8$.

Kopteva's estimate (18) cannot be applied to (26). We use this problem to compare our two estimates (20) and (21). We see in Table 4 that they are relatively close, (21) being somewhat worse, as should be expected. The same conclusion applies to Table 5 which contains the results for the nonlinear problem (27). In this example, our estimates cannot compete with Δ, but the comparison is not fair. The low efficiency of (20) and (21) is mainly caused by the large value of the coefficient $2/\beta$ since $\beta = \exp(-\pi/2)$ ($\bar{\beta}$ needed in (21) is simply 1), see [3]. Note that this difficulty is not present in problems (25) and (26), where $\beta = \bar{\beta} = 1$. In order to illustrate the influence of the coefficient $2/\beta$, we include in Table 5 the quantity

$$\Delta^* = \frac{2}{\beta}\Delta.$$

The efficiency of Δ^* is in fact worse than that of the estimates in (20) and (21).

N	E Ord	(20) Ord	Eff	(21) Ord	Eff	Δ Ord	Eff
32	1.73E−1	4.25E−1	.41	4.76E−1	.36	3.67E−1	.47
	.95	1.00		1.07		.90	
64	8.91E−2	2.12E−1	.42	2.27E−1	.39	1.97E−1	.45
	.98	1.00		1.05		.95	
128	4.52E−2	1.06E−1	.43	1.10E−1	.41	1.02E−1	.44
	.99	1.00		1.02		.97	
256	2.28E−2	5.31E−2	.43	5.41E−2	.42	5.21E−2	.44
	.99	1.00		1.01		.99	
512	1.14E−2	2.65E−2	.43	2.68E−2	.43	2.63E−2	.43

Table 4. Errors, error estimates, their numerical orders of convergence, and error-estimate efficiency values for (26) solved on X_B^N with $q = 0.8$.

N	E Ord	(20) Ord	Eff	(21) Ord	Eff	Δ Ord	Eff	Δ^* Ord	Eff
32	1.26E−1	1.18E+0	.11	1.53E+0	.08	1.81E−1	.69	1.74E+0	.07
	.94	1.00		1.13		.80		.80	
64	6.53E−2	5.90E−1	.11	6.99E−1	.09	1.04E−1	.63	1.00E+0	.07
	.97	1.00		1.10		.89		.89	
128	3.34E−2	2.95E−1	.11	3.26E−1	.10	5.63E−2	.59	5.41E−1	.06
	.98	1.00		1.06		.94		.94	
256	1.69E−2	1.48E−1	.11	1.56E−1	.11	2.93E−2	.58	2.82E−1	.06
	.99	1.00		1.04		.97		.97	
512	8.50E−3	7.38E−2	.12	7.60E−2	.11	1.50E−2	.57	1.44E−1	.06

Table 5. Errors, error estimates, their numerical orders of convergence, and error-estimate efficiency values for (27) solved on X_B^N with $q = 0.8$.

References

[1] J. Lorenz, Stability and monotonicity properties of stiff quasilinear boundary problems, *Univ. Novom Sadu Zb. Rad. Prir.-Mat. Fak. Ser. Mat.*, **12**, 151–175 (1982).

[2] R. Vulanović, A uniform numerical method for quasilinear singular perturbation problems without turning points, *Computing*, **41**, 97–106 (1989).

[3] R. Vulanović, A priori meshes for singularly perturbed quasilinear two-point boundary value problems, *IMA J. Numer. Anal.*, **21**, 349–366 (2001).

[4] K.W. Chang and F.A. Howes, *Nonlinear Singular Perturbation Phenomena: Theory and Application*, Springer, New York (1984).

[5] J.J.H. Miller (Editor), *Applications of Advanced Computational Methods for Boundary and Interior Layers*, Boole Press, Dublin (1993).

[6] T. Linß, H.-G. Roos, and R. Vulanović, Uniform pointwise convergence on Shishkin-type meshes for quasilinear convection-diffusion problems, *SIAM J. Numer. Anal.*, **38**, 897–912 (2000).

[7] N. Kopteva, Maximum norm a posteriori error estimates for a one-dimensional convection-diffusion problem, *SIAM J. Numer. Anal.*, **39**, 423–441 (2001).

[8] T. Linß, Sufficient conditions for uniform convergence on layer-adapted grids, *Appl. Numer. Math.*, **37**, 241–255 (2001).

[9] R. Vulanović, On a numerical solution of a type of singularly perturbed boundary value problem by using a special discretization mesh, *Univ. Novom Sadu Zb. Rad. Prir.-Mat. Fak. Ser. Mat.*, **13**, 187–201 (1983).

[10] L. Abrahamsson and S. Osher, Monotone difference schemes for singular perturbation problems, *SIAM J. Numer. Anal.*, **19**, 979–992 (1982).

[11] J. Lorenz, Combinations of initial and boundary value methods for a class of singular perturbation problems. In *Proceedings of the Conference on the Numerical Analysis of Singular Perturbation Problems*, (Edited by P.W. Hemker and J.J.H. Miller), pp. 295–315, Academic Press, London, (1979).

[12] R. Vulanović, A second order numerical method for nonlinear singular perturbation problems without turning points, *Zh. Vychisl. Mat. Mat. Fiz.*, **31**, 522–532 (1991).

[13] R.E. O'Malley, *Introduction to Singular Perturbations*, Academic Press, New York, (1974).