ADIABATIC INVARIANTS FOR N CONNECTED LINEAR OSCILLATORS

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SUMMARY

It is known that the ratio of energy and frequency is an adiabatic invariant for linear harmonic oscillator. In the paper some new adiabatic invariants are found for linear ordinary differential equation of order $2n$ corresponding to n connected oscillators. The denominators of these invariants are vanishing when differences of some frequencies tend to zero (resonance). The changes of the considered invariants are estimated.

§1. MAIN RESULTS

Let $x(t, \varepsilon)$ be a solution of the equation of linear harmonic oscillator:

$$D_t^2 x + \omega^2(\varepsilon t)x = 0, \quad t \in R,$$

where $D_t = d/dt$ and $\varepsilon > 0$ is an arbitrarily small parameter. The ratio of energy and frequency

$$J(t, \varepsilon) = \frac{[D_t x]^2 + \omega^2 x^2}{2\omega} = \frac{E}{\omega} \tag{1.0}$$

is called adiabatic invariant and the full change of $J(t, \varepsilon)$ can be estimated as $J(\varepsilon) = J(\infty, \varepsilon) - J(-\infty, \varepsilon) = O(\varepsilon^\infty)$, $\varepsilon \to 0$ if ω and x satisfy some conditions (see for instance [1], [3], [4]).

Consider the ordinary linear differential equation

$$D_t^{2n} x + \sum_{k=1}^{n} \sum_{1 \le i_1 < \ldots < i_k \le n} \omega_{i_1}^2 \ldots \omega_{i_k}^2 D_t^{2(n-k)} x = 0, \tag{1.1}$$

where $D_t = \frac{d}{dt}$, $\omega_m = \omega_m(\varepsilon t)$, $m = 1, \ldots, n$ and $\varepsilon > 0$ is an arbitrarily small parameter.

Everywhere below we shall assume that the frequencies $\{\omega_m(\tau)\}_{m=1}^n$, where $\tau = \varepsilon t$, satisfy the following conditions:

(i) $\omega_m(\tau) \in C^{2n}(R)$, the functions $\omega_m(\tau)$ are positive and different;

(ii) $\omega_m(\tau)$ have finite, positive and different limits $\omega_m(\pm\infty) = \omega_m^\pm$;

(iii) $\int\limits_{-\infty}^{\infty} | \omega_m^{(k)}(\tau) | \, d\tau < \infty, \quad k = 1, 2, \ldots, 2n.$

Let $\varphi_j(t,\varepsilon)$, $j=1,\dots,2n$ be linearly independent, $2n$ times continuosly differentiable by $t\in R$ asymptotic solutions of (1.1), i.e.

$$D_t^{k-1}\psi_j=[1+\varepsilon\delta_{jk}(t,\varepsilon)]D_t^{k-1}\varphi_j,\quad j,k=1,\dots,2n,$$

where $\{\psi_j(t,\varepsilon)\}_1^{2n}$ is a fundamental system of solutions of (1.1) and $\mid\delta_{jk}(t,\varepsilon)\mid\le c$ for any $t\in R$, $\varepsilon>0$.

Definition 1. *The quantities*

$$J_k(t,\varepsilon)=I_k(t,x,\varepsilon)=\left|\frac{W(\varphi_1,\dots,\varphi_{k-1},x,\varphi_{k+1},\dots,\varphi_{2n})}{W(\varphi_1,\dots,\varphi_{2n})}\right|^2,$$

$k=1,\dots,m$, *we call adiabatic invariants of (1.1). Here* $x=x(.,\varepsilon)\in C^m(R)$ *is a solution of (1.1) with bounded by* ε *Cauchy data* $x(0,\varepsilon)=x_0,\dots,D_t^m x(0,\varepsilon)=x_m$, $W(\varphi_1,\dots,\varphi_{2n})=\det(D_t^{k-1}\varphi_j)_{k,j=1}^{2n}$ *is the Wronskian of* $\varphi_1,\dots,\varphi_{2n}$.

Taking $\varphi_{12}=\omega^{-1/2}\exp(\pm i\int\limits_0^t\omega(\varepsilon s)ds)$ one can obtain the classical formula (1.0) for the equation of oscillator.

Denote

$$P_m=D_t^{2n-2}+\sum_{k=1}^{n-1}\sum_{1\le i_1<\dots<i_k\le n,m\notin\{i_p\}_{p=1}^k}\omega_{i_1}^2\dots\omega_{i_k}^2D_t^{2(n-k-1)},$$

$$K_{ml}(\tau)=\omega_m^2(\tau)-\omega_l^2(\tau);\quad K=\prod_{1\le m<l\le n}K_{ml},\quad E_m=(P_m(D_tx))^2+\omega_m^2(P_mx)^2.$$

One can observe that the quantities E_m are first integrals of (1.1), i.e. $D_tE_m=0$ for $\omega_m=\text{const}$. Substituting in P_m $\tau=\varepsilon t$, we obtain the operator

$$L_m=\varepsilon^{2n-2}D_\tau^{2n-2}+\sum_{k=1}^{n-1}\sum_{1\le i_1<\dots<i_k\le n,m\notin\{i_p\}_{p=1}^k}\omega_{i_1}^2\dots\omega_{i_k}^2\varepsilon^{2(n-k-1)}D_\tau^{2(n-k-1)}.$$

In this paper we show (§3, Theorem 2) that if the conditions **(i)** – **(iii)** are satisfied, then there exist adiabatic invariants of (1.1) which have the forms

$$J_m(t,\varepsilon)=\frac{E_m}{\omega_m}\exp\left(2\int\limits_0^{\varepsilon t}\sum_{l=1,l\ne m}^n\frac{D_\tau[\omega_l^2(\tau)]}{K_{ml}(\tau)}d\tau\right)\qquad m=1,\dots,n\qquad(1.2)$$

and satisfy the following conditions:

1° there exist some constants $C > 0$ and $\varepsilon' > 0$ such that for any $t_1, t_2 \in (-\infty, \infty)$ and $0 < \varepsilon < \varepsilon'$

$$| J_k(t_1, \varepsilon) - J_k(t_2, \varepsilon) | \leq C\varepsilon; \tag{1.3}$$

2° $$J_k(\varepsilon) = J_k(\infty, \varepsilon) - J_k(-\infty, \varepsilon) = O(\varepsilon), \quad \text{as} \quad \varepsilon \to 0, \quad k = 0, \cdots, n \tag{1.4}$$

Example. For forth order equation

$$D_t^4 x + [\omega_1^2(\varepsilon t) + \omega_2^2(\varepsilon t)] D_t^2 x + \omega_1^2(\varepsilon t)\omega_2^2(\varepsilon t) x = 0 \tag{1.5}$$

we have

$$J_1(t, \varepsilon) = \frac{E_1}{\omega_1} \exp\left(2 \int_0^t \frac{D_t[\omega_2^2(\varepsilon t)]}{K} dt\right), \tag{1.6}$$

$$J_2(t, \varepsilon) = \frac{E_2}{\omega_2} \exp\left(-2 \int_0^t \frac{D_t[\omega_1^2(\varepsilon t)]}{K} dt\right), \tag{1.7}$$

where $E_k = (D_t^3 x + \omega_{3-k}^2 D_t x)^2 + \omega_k^2 (D_t^2 x + \omega_{3-k}^2 x)^2$, $K = \omega_1^2 - \omega_2^2$.
In [7] we also show that the full changes of $J_k(t, \varepsilon)$, $J_k(\varepsilon) = J_k(\infty, \varepsilon) - J_k(-\infty, \varepsilon)$ are exponentially small if ω_k satisfy some additional conditions of holomorphity.

§2. ASYMPTOTIC SOLUTIONS OF EQUATION (1.1)

One can rewrite (1.1) in the form

$$Lu = D_\tau^{2n} u + \sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \omega_{i_1}^2 \ldots \omega_{i_k}^2 \varepsilon^{-2k} D_\tau^{2(n-k)} u = 0, \tag{2.1}$$

where $\tau = \varepsilon t$ and $u(\tau, \varepsilon) = x(t, \varepsilon)$. Solutions of this equation can be expressed by the roots $\pm(i\omega_m)/\varepsilon$ of the coresponding characteristic equation

$$\lambda^{2n} + \sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \omega_{i_1}^2 \ldots \omega_{i_k}^2 \varepsilon^{-2k} \lambda^{2(n-k)} = 0.$$

In virtue of a generalization of Levinson asymptotic theorem ([7], theorem 2) from **(i)** – **(iii)** it follows that for $\tau \in [T^+, \infty)$ there exist some solutions $\{u_j^+(\tau, \varepsilon)\}_{j=1}^{2n} \in C_\tau^{2n}([T^+, \infty))$ of (2.1), representable in the form

$$D_\tau^{k-1} u_j^+ = D_\tau^{k-1} \widetilde{u}_j [1 + h_{jk}^+(\tau, \varepsilon)], \quad (1 \leq j, k \leq 2n),$$

where

$$\widetilde{u}_{2m-1} = \omega_m^{-1/2} \exp \int_0^\tau \left(\frac{i\omega_m(t)}{\varepsilon} - \sum_{l=1, l\neq m}^{n} \frac{D_t[\omega_m^2(t)]}{K_{ml}(t)} \right) dt,$$

$$\widetilde{u}_{2m} = \bar{\widetilde{u}}_{2m-1}, \qquad m = 1, \ldots, n.$$

The functions $h^+_{jk}(\tau, \varepsilon)$ for some $c > 0$ satisfy the estimate

$$c \mid h^+_{jk}(\tau, \varepsilon) \mid \leq \max_{1 \leq s,q \leq 2n} \left\{ \exp \int_\tau^\infty \mid \Phi^{2n}_s L\widetilde{u}_q(t) W^{-1}(t, \widetilde{u}_1, \ldots, \widetilde{u}_{2n}) \mid dt \right\} - 1.$$

Here $W(\tau, \widetilde{u}_1, \ldots, \widetilde{u}_{2n}) = \det \Phi$,

$$\Phi = \begin{pmatrix} \widetilde{u}_1 & \ldots & \widetilde{u}_{2n} \\ \vdots & \ddots & \vdots \\ D^{2n-1}_\tau \widetilde{u}_1 & \ldots & D^{2n-1}_\tau \widetilde{u}_{2n} \end{pmatrix},$$

and Φ^{2n}_s are the minors of Φ obtained by deleting its $2n$-th line and s-th column. One can be convinced that from **(iii)** it follows that

$$\lim_{\tau \to \pm\infty} D^k_\tau \omega_m(\tau) = 0, \quad k = 1, \ldots, 2n-1. \tag{2.2}$$

Indeed, as

$$D_\tau \omega_1 = \int_a^\tau D^2_s \omega_1(s) ds + D_\tau \omega_1(a),$$

the limits $D_\tau \omega_1(\pm\infty)$ are finite. On the other hand, there exists $\int\limits_{-\infty}^{+\infty} D_s \omega_1 ds$. Therefore $D_\tau \omega_1(\pm\infty) = 0$ (see [8]). The same argument completes the proof of (2.2).

A direct calculation and use of (4.2) gives

$$K(0)^{-2} \varepsilon^{(2n-1)n} W(\tau, \widetilde{u}_1, \ldots, \widetilde{u}_{2n}) = (-2i)^n + \varepsilon \rho(\tau, \varepsilon),$$

$$\mid L\widetilde{u}_q \mid \leq \frac{C}{\varepsilon^{2n-2}}, \qquad \mid \Phi^{2n}_s \mid \leq \frac{C}{\varepsilon^{(n-1)(2n-1)}},$$

where $\lim\limits_{\tau \to \pm\infty} \rho(\tau, \varepsilon) = 0$ and $\mid \rho(\tau, \varepsilon) \mid \leq C$. Therefore the solutions $\{u^+_j\}^{2n}_{j=1}$ take the form

$$D^{k-1}_\tau u^+_j = D^{k-1}_\tau \widetilde{u}_j [1 + \varepsilon \rho^+_{jk}(\tau, \varepsilon)], \qquad (1 \leq j, k \leq 2n), \tag{2.3}$$

where $\lim\limits_{\tau \to +\infty} \rho^+_{jk}(\tau, \varepsilon) = 0$ and $\mid \rho^+_{jk}(\tau, \varepsilon) \mid \leq C$ for any $\varepsilon > 0$ and $\tau \in [T^+, \infty)$.

Also there exist solutions $\{u^-_j\}^{2n}_{j=1} \in C^{2n}((-\infty, T^-])$ of the equation (2.1) which for $\tau \in (-\infty, T^-]$ can be represented in the form

$$D^{k-1}_\tau u^-_j = D^{k-1}_\tau \widetilde{u}_j [1 + \varepsilon \rho^-_{jk}(\tau, \varepsilon)], \qquad (1 \leq j, k \leq 2n), \tag{2.3'}$$

where $\lim_{\tau\to-\infty} \rho^-_{jk}(\tau,\varepsilon) = 0$ and $\mid \rho^-_{jk}(\tau,\varepsilon) \mid\leq C(T)$ for any $\varepsilon > 0$ and $\tau \in (-\infty, T]$. Assuming that u is a solution of (2.1) on $[T^-, T^+]$ observe that some functions σ_j are uniquely defined by formulas

$$D_\tau^{k-1} u = \sum_{j=1}^{2n} \sigma_j D_\tau^{k-1} \widetilde{u}_j \quad k = 1, \cdots, 2n.$$

Hence we get the following system

$$\begin{cases} \sum_{j=1}^{2n} D_\tau \sigma_j D_\tau^{m-1} \widetilde{u}_j = 0, & m = 1, \cdots, 2n-1, \\ \sum_{j=1}^{2n} D_\tau \sigma_j D_\tau^{2n-1} \widetilde{u}_j = -\sum_{j=1}^{2n} \sigma_j L \widetilde{u}_j. \end{cases}$$

The solutions of this system are

$$D_\tau \sigma_k = \sum_{j=1}^{2n} B_{jk} \sigma_j, \quad \text{where} \quad B_{jk} = -W^{-1}(\widetilde{u}_1, \cdots, \widetilde{u}_{2n}) L \widetilde{u}_j \Phi_k^{2n}.$$

Integration of $D_\tau \sigma_k$ gives

$$\sigma_k = C_k + \int_0^\tau \sum_{j=1}^{2n} B_{jk} \sigma_j ds, \quad k = 1, \ldots, 2n.$$

Using the obvious estimate

$$\sum_{j=1}^{2n} \mid \sigma_j \mid\leq \sum_{j=1}^{2n} \mid C_j \mid + 2n \int_0^\tau B \sum_{j=1}^{2n} \mid \sigma_j \mid ds,$$

where $B(s) = \max_{ij} \mid B_{ij}(s) \mid\leq C\varepsilon$ and the well known Gronwall's lemma we get

$$\sum_{j=1}^{2n} \mid \sigma_j \mid\leq \sum_{j=1}^{2n} \mid C_j \mid \exp\left(2n \int_0^\tau B\, ds\right).$$

Consequently

$$\varepsilon \mid \delta_k \mid\leq \frac{1}{2n} \sum_{j=1}^{2n} \mid C_j \mid \left(\exp\left(2n \int_0^\tau B\, ds\right) - 1\right),$$

where $\sigma_k = C_k + \varepsilon\delta_k$. From the conditions

$$D_\tau^{k-1} u(T^+, \varepsilon) = D_\tau^{k-1} u_j^+(T^+, \varepsilon), \quad k = 1, \cdots, 2n$$

it follows that

$$C_k + \varepsilon\delta_k(T^+,\varepsilon) = \begin{cases} 1+\varepsilon a_j, & k=j, \\ \varepsilon a_k, & k\neq j, \end{cases} \quad \{a_i\}_{i=1}^{2n} = \text{const}$$

and

$$\sum_{k=1}^{2n} \mid C_k \mid \left(2 - \exp\left(2n\int\limits_0^{T^+} B\,ds\right)\right) \leq$$

$$\leq \sum_{k=1}^{2n} \big(\mid C_k \mid -\varepsilon \mid \delta_k(T^+,\varepsilon) \mid\big) \leq \left|1+\varepsilon\sum_{k=1}^{2n} a_k\right|.$$

Therefore C_k and δ_k are bounded by $\varepsilon > 0$ and

$$D_\tau^{k-1}u = D_\tau^{k-1}\widetilde{u}_j[1+\varepsilon\rho_{jk}(\tau,\varepsilon)], \quad \mid \rho_{jk}(\tau,\varepsilon) \mid\leq C.$$

By same way one can extend the solutions (2.3) on $(-\infty, T^-]$. So we proved the following

Theorem 1. *The equation (2.1) has solutions* $u_j^\pm \in C_\tau^{2n}(R)$, $j = 1,\cdots,2n$ *of the form (2.3), (2.3'), where* $\lim\limits_{\tau\to\pm\infty}\rho_{jk}^\pm(\tau,\varepsilon) = 0$ *and* $\mid \rho_{jk}^\pm \mid< C$ *for any* $\varepsilon > 0$ *and* $\tau \in (-\infty,\infty)$.

As one can verify

$$K(0)^{-2}\varepsilon^{(2n-1)n}W^\pm = (-2i)^n + \circ(1), \quad \tau\to\pm\infty. \tag{2.4}$$

Here $W^\pm = W(\tau, u_1^\pm,\ldots,u_{2n}^\pm) = \det\Phi^\pm$ and

$$\Phi^\pm = \begin{matrix} u_1^\pm & \ldots & u_{2n}^\pm \\ \vdots & \ddots & \vdots \\ D_\tau^{2n-1}u_1^\pm & \ldots & D_\tau^{2n-1}u_{2n}^\pm \end{matrix}.$$

In theory of ordinary differential equations it is well known that $W^\pm$ is independent of τ. Therefore as $\tau\to\pm\infty$ (2.4) takes the form

$$K(0)^{-2}\varepsilon^{(2n-1)n}W^\pm = (-2i)^n.$$

Since $\{u_j^+\}_{j=1}^{2n}$ and $(\{u_j^-\}_{j=1}^{2n})$ are the fundamental solutions of (2.1), any solution u of (2.1) is representable in the form

$$u = a_1^+u_1^+ + \ldots + a_{2n}^+u_{2n}^+ = a_1^-u_1^- + \ldots + a_{2n}^-u_{2n}^-, \tag{2.5}$$

where $\{a_j^\pm\}_{j=1}^{2n}$ depend on ε.

Introduce a matrix

$$\alpha(\varepsilon) = \begin{matrix} \alpha_{1,1} & \dots & \alpha_{1,2n} \\ \vdots & \ddots & \vdots \\ \alpha_{2n,1} & \dots & \alpha_{2n,2n} \end{matrix} \qquad (\alpha_{i,j} = \alpha_{i,j}(\varepsilon))$$

by the formula

$$u^- = \alpha u^+, \qquad \text{where} \qquad u^\pm = \text{colon}(u_1^\pm, \dots, u_{2n}^\pm). \tag{2.6}$$

Using the definition of $\alpha(\varepsilon)$ and the obvious equalities $\det \Phi^- = \det \Phi^+$,

$$\Phi^{-T} = (u^-, \dots, D_\tau^{2n-1} u^-) = \alpha(u^+, \dots, D_\tau^{2n-1} u^+) = \alpha \Phi^{+T}$$

and $\Phi^- = \Phi^+ \alpha^T$, where $u^\pm, \dots, D_\tau^{2n-1} u^\pm$ are vector-columns, we obtain $\det \alpha = 1$.

Using (2.5) one can easily show that

$$a^+ = \alpha^T a^-, \text{where} \quad a^\pm = \text{colon}(a_1^\pm, \dots, a_{2n}^\pm). \tag{2.7}$$

Lemma 1. *If the conditions* **(i)** – **(iii)** *are satisfied, then*

$$\alpha_{ij} = \delta_{ij} + O(\varepsilon) \tag{2.8}$$

Proof. Differentiating (2.6) k times ($0 \leq k \leq 2n-1$) and solving the obtained systems by α_{ij} we get

$$\alpha_{ij} = \frac{W(\tau, u_1^+, \dots, u_{j-1}^+, u_i^-, u_{j+1}^+, \dots, u_{2n}^+)}{W^+}$$

and

$$\alpha_{ij} = \frac{W(\tau, \widetilde{u}_1, \dots, \widetilde{u}_{j-1}, \widetilde{u}_i, \widetilde{u}_{j+1}, \dots, \widetilde{u}_{2n})}{W^+} + \varepsilon \kappa_{ij}(\tau, \varepsilon), \tag{2.9}$$

where $1 \leq i, j \leq 2n$ and $\mid \kappa_{ij}(\tau, \varepsilon) \mid \leq C$. Now (2.8) easily follows from (2.9). Consequently, if $\{a_j^-\}_{j=1}^{2n}$ are bounded in some interval $(0, \varepsilon_0)$, then $\{a_j^+\}_{j=1}^{2n}$ are bounded in other interval $(0, \varepsilon_1)$.

§3. ESTIMATES OF CHANGES OF ADIABATIC INVARIANTS

Lemma 2. *If the conditions* **(i)** – **(iii)** *are satisfied and the solution of (2.1) is representable in the form (2.5), where* $\{a_j^-\}_{j=1}^{2n}$ *are bounded in* $(0, \varepsilon_0)$*, then*

$$4Q_m(0)^{-2} J_m(\varepsilon) = a_{2m-1}^+ a_{2m}^+ - a_{2m-1}^- a_{2m}^-, \quad m = 1, \dots, n, \tag{3.1}$$

where $Q_m = \prod K_{ml}$*,* $1 \leq l \leq n$*,* $l \neq m$*, and* $J_m(\varepsilon)$ *are defined by (1.4).*

Proof. Differentiating (2.5) k times $(0 \le k \le 2n-1)$ and solving the obtained system by a_k we get $W_k^{\pm} = a_k^{\pm}$, where

$$W_k^{\pm} = (-2i)^{-n} K(0)^{-2} \varepsilon^{(2n-1)n} W(\tau, u_1^{\pm}, \dots, u_{k-1}^{\pm}, u, u_{k+1}^{\pm}, \dots, u_{2n}^{\pm})$$

. On the other hand, using the boundedness of $\{a_j^{\pm}\}_{j=1}^{2n}$ and (4.4) we obtain the folowing two formulas:

$$W_{2m-1}^{\pm} = (-1)^{n-1} \frac{\varepsilon L_m D_\tau u + i\omega_m L_m u}{2iQ_m(0)\omega_m^{1/2}} \times$$

$$\times \exp \int_0^\tau \left(-\frac{i\omega_m(t)}{\varepsilon} + \sum_{l=1, l\neq m}^{n} \frac{D_t[\omega_l^2(t)]}{K_{ml}(t)} \right) dt + \varepsilon \rho_{2m-1}^{\pm}(\tau, \varepsilon),$$

$$W_{2m}^{\pm} = (-1)^n \frac{\varepsilon L_m D_\tau u - i\omega_m L_m u}{2iQ_m(0)\omega_m^{1/2}} \exp \int_0^\tau \left(\frac{i\omega_m(t)}{\varepsilon} + \sum_{l=1, l\neq m}^{n} \frac{D_t[\omega_l^2(t)]}{K_{ml}(t)} \right) dt + \varepsilon \rho_{2m}^{\pm}(\tau, \varepsilon),$$

where $\lim\limits_{\tau \to \pm\infty} \rho_m^{\pm}(\tau, \varepsilon) = 0$ and $|\ \rho_m^{\pm}(\tau, \varepsilon)\ | \le C$. Thus, if the solution has the form (2.5), where $\{a_j^-\}_{j=1}^{2n}$ are bounded, then

$$\frac{J_m(t,\varepsilon)}{4Q_m(0)^2} = a_{2m-1}^- a_{2m}^- + \varepsilon \widetilde{\rho}_m^-(t,\varepsilon) = a_{2m-1}^+ a_{2m}^+ + \varepsilon \widetilde{\rho}_m^+(t,\varepsilon), \tag{3.2}$$

where $\lim\limits_{t \to \pm\infty} \rho_m^{\pm}(t, \varepsilon) = 0$ and $|\ \rho_m^{\pm}(t, \varepsilon)\ | \le C$. From (3.2) easily imply (3.1).

Theorem 2. *If the conditions* **(i)** – **(iii)** *are fulfilled, then there exist adiabatic invariants of (1.1) which have the forms (1.2) and satisfy the estimates (1.3) and (1.4).*

Proof. Let $x = x(.,\varepsilon) \in C^{2n}(R)$ be a solution of (1.1) with bounded Caucy data. Then from the formula $W_k^{\pm} = a_k^{\pm}$ it will follow that in the representation (2.5) the coefficients $a_k^{\pm}$ are bounded by ε. From (2.7) and (3.1) it follows that

$$4Q_m(0)^{-2} J_m(\varepsilon) = \sum_{j=1}^{2n} \alpha_{j,2m-1} a_j^- \sum_{j=1}^{2n} \alpha_{j,2m} a_j^- - a_{2m-1}^- a_{2m}^-, m = 1, \dots, n.$$

¿From (2.8) we have $J_m(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$.

From (3.2) it follows that

$$|\ J_m(t_1, \varepsilon) - J_m(t_2, \varepsilon)\ | \le C\varepsilon.$$

The theorem is proved.

§4. PROOFS OF AUXILIARY FORMULAS

Let $(a_{ij})^{2n+1}$ be a $(2n+1)\times(2n+1)$ matrix. Assume that one of the cases

a) $a_{ij}=0$ when $i+j$ is even,

b) $a_{ij}=0$ when $i+j$ is odd,

is valid. Denoting $b_{ij}=a_{2i-1,2j-1}$, $c_{ij}=a_{2i,2j}$ and

$$A_{2n+1}=\det(a_{ij})^{2n+1},\quad B_{n+1}=\det(b_{ij})^{n+1},\quad C_n=\det(c_{ij})^n$$

we shall prove the formula

$$A_{2n+1}=\begin{cases}0, & \text{if a) is valid,}\\ B_{n+1}C_n, & \text{if b) is valid.}\end{cases}\tag{4.1}$$

The first line is true, i.e. the determinant is zero because $a_{1,i_1}a_{2,i_2}\dots a_{2n+1,i_{2n+1}}=0$. Indeed, these products differ from zero only if the quantities i_{2k+1} $(k=0,\dots,n)$ are even. But the number of even columns of the determinant is n. Therefore $a_{1,i_1}a_{2,i_2}\dots a_{2n+1,i_{2n+1}}=0$. The second line of (4.1) can be proved by induction. Indeed

$$\begin{vmatrix}a_{11} & 0 & a_{13}\\ 0 & a_{22} & 0\\ a_{31} & 0 & a_{33}\end{vmatrix}=a_{22}\begin{vmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}.$$

If $A_{2n-1}=B_nC_{n-1}$ and $\{a_{1,2j-1}\}_{j=1}^{n+1},\{a_{2,2j}\}_{j=1}^{n}\neq 0$, then

$$A_{2n+1}=a_{11}\dots a_{1,2n+1}a_{22}\dots a_{2,2n}\begin{vmatrix}\frac{a_{33}}{a_{13}}-\frac{a_{31}}{a_{11}} & \dots & \frac{a_{3,2n+1}}{a_{1,2n+1}}-\frac{a_{31}}{a_{11}}\\ \vdots & \ddots & \vdots\\ \frac{a_{2n+1,3}}{a_{13}}-\frac{a_{2n+1,1}}{a_{11}} & \dots & \frac{a_{2n+1,2n+1}}{a_{1,2n+1}}-\frac{a_{2n+1,1}}{a_{11}}\end{vmatrix}\times$$

$$\times\begin{vmatrix}\frac{a_{44}}{a_{24}}-\frac{a_{42}}{a_{22}} & \dots & \frac{a_{4,2n}}{a_{2,2n}}-\frac{a_{42}}{a_{22}}\\ \vdots & \ddots & \vdots\\ \frac{a_{2n,4}}{a_{24}}-\frac{a_{2n,2}}{a_{22}} & \dots & \frac{a_{2n,2n}}{a_{2,2n}}-\frac{a_{2n,2}}{a_{22}}\end{vmatrix}=$$

$$=\begin{vmatrix}a_{11} & a_{13} & \dots & a_{1,2n+1}\\ a_{31} & a_{33} & \dots & a_{3,2n+1}\\ \vdots & \vdots & \ddots & \vdots\\ a_{2n+1,1} & a_{2n+1,3} & \dots & a_{2n+1,2n+1}\end{vmatrix}\begin{vmatrix}a_{22} & a_{24} & \dots & a_{2,2n}\\ a_{42} & a_{44} & \dots & a_{4,2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{2n,2} & a_{2n,4} & \dots & a_{2n,2n}\end{vmatrix}.$$

One can easily prove the last formula without assuming that $\{a_{1,2j-1}\}_{j=1}^{n+1},\{a_{2,2j}\}_{j=1}^{n}\neq 0$.

We shall use the following well-known formula for the Van der Mond determinant

$$\begin{vmatrix}1 & 1 & \dots & 1\\ \alpha_1 & \alpha_2 & \dots & \alpha_m\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_1^{m-1} & \alpha_2^{m-1} & \dots & \alpha_m^{m}\end{vmatrix}=\prod_{1\le i<j\le m}(\alpha_j-\alpha_i).\tag{4.2}$$

By induction one can prove that

$$\begin{vmatrix} f_0 & 1 & \dots & 1 \\ f_1 & \alpha_1 & \dots & \alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ f_m & \alpha_1^m & \dots & \alpha_m^m \end{vmatrix} =$$

$$= (-1)^m \prod_{1\le i<j\le m} (\alpha_j - \alpha_i)[f_m + \sum_{k=1}^{m} (-1)^k f_{m-k} \sum_{1\le i_1<\dots<i_k\le m} \alpha_{i_1}\dots\alpha_{i_k}]. \qquad (4.3)$$

Now we can calculate the following determinant:

$$A = \begin{vmatrix} u_0 & 1 & 1 & 1 & \dots & 1 & 1 \\ u_1 & -x_1 & x_2 & -x_2 & \dots & x_n & -x_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{2n-2} & x_1^{2n-2} & x_2^{2n-2} & x_2^{2n-2} & \dots & x_n^{2n-2} & x_n^{2n-2} \\ u_{2n-1} & -x_1^{2n-1} & x_2^{2n-1} & -x_2^{2n-1} & \dots & x_n^{2n-1} & -x_n^{2n-1} \end{vmatrix}.$$

The suitable elementary transformations give

$$A = 2^{n-1} x_2 \dots x_n \begin{vmatrix} u_0 & 1 & 0 & 1 & \dots & 0 & 1 \\ u_1 + x_1 u_0 & 0 & 1 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{2n-2} & x_1^{2n-2} & 0 & x_2^{2n-2} & \dots & 0 & x_n^{2n-2} \\ u_{2n-1} + x_1 u_{2n-2} & 0 & x_2^{2n-2} & 0 & \dots & x_n^{2n-2} & 0 \end{vmatrix}.$$

Using the first formula of (4.1) we get

$$A = -2^{n-1} x_2 \dots x_n \sum_{k=1}^{n} (-1)^{n-k} \times$$

$$\times \begin{vmatrix} 1 & 0 & 1 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_2^{2k-4} & 0 & \dots & x_n^{2k-4} & 0 \\ x_1^{2k-2} & 0 & x_2^{2k-2} & \dots & 0 & x_n^{2k-2} \\ 0 & x_2^{2k} & 0 & \dots & x_n^{2k} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{2n-2} & 0 & x_2^{2n-2} & \dots & 0 & x_n^{2n-2} \end{vmatrix} (u_{2k-1} + x_1 u_{2k-2}).$$

Now using (4.2) and the second formula of (4.1) we obtain

$$A = (-1)^n 2^{n-1} x_2 \dots x_n \prod_{1\le i<j\le n} (x_j^2 - x_i^2) \sum_{k=1}^{n} (-1)^{k+1} \times$$

$$
\times \begin{vmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ x_2^{2k-4} & \dots & x_n^{2k-4} \\ x_2^{2k} & \dots & x_n^{2k} \\ \vdots & \ddots & \vdots \\ x_2^{2n-2} & \dots & x_n^{2n-2} \end{vmatrix} (u_{2k-1} + x_1 u_{2k-2})
$$

and

$$
A = (-1)^n 2^{n-1} x_2 \dots x_n \prod_{1 \le i < j \le n} (x_j^2 - x_i^2) \begin{vmatrix} u_1 + x_1 u_0 & 1 & \dots & 1 \\ u_3 + x_1 u_2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ u_{2n-1} + x_1 u_{2n-2} & x_2^{2n-2} & \dots & x_n^{2n-2} \end{vmatrix}.
$$

At last, from (4.3) it follows that

$$
A = -2^{n-1} x_2 \dots x_n \prod_{1 \le i < j \le n} (x_j^2 - x_i^2) \prod_{2 \le i < j \le n} (x_j^2 - x_i^2) \times
$$

$$
\times [u_{2n-1} + x_1 u_{2n-2} + \sum_{k=1}^{n-1} ((-1)^k u_{2(n-k)-1} + x_1 u_{2(n-k)-2}) \sum_{2 \le i_1 < \dots i_k \le n} x_{i_1}^2 \dots x_{i_k}^2]. \quad (4.4)
$$

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