A dual version of Huppert's conjecture on conjugacy class sizes *

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Abstract

In [1], a conjecture of J.G. Thompson for $\mathrm{PSL}_n(q)$ was proved. It was shown that every finite group G with the property $\mathbf{Z}(G) = 1$ and $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_n(q))$ is isomorphic to $\mathrm{PSL}_n(q)$ where $\mathrm{cs}(G)$ is the set of conjugacy class sizes of G. In this article we improve this result for $\mathrm{PSL}_2(q)$. In fact we prove that if $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_2(q))$, for $q > 3$, then $G \cong \mathrm{PSL}_2(q) \times A$, where A is abelian. Our proof does not depend on the classification of finite simple groups.

keywords: Conjugacy classes, Simple groups, Huppert's conjecture.

1 Introduction

Let G be a finite group and $\mathbf{Z}(G)$ be its center. For $x \in G$, assume that x^G is the conjugacy class of G containing x and $C_G(x)$ denotes the centralizer of x in G. We use $\mathrm{cs}(G)$ for the set of all conjugacy class sizes of G, that is $\mathrm{cs}(G) = \{|g^G| : g \in G\}$. In this article we are concerned with the following open conjecture (Conjecture 1):

Conjecture 1. (Dual of Huppert's Conjecture). *Let G be a finite group and let H be a nonabelian simple group. If $\mathrm{cs}(G) = \mathrm{cs}(H)$, then $G \cong H \times A$, where A is abelian.*

*Mathematics Subject Classification: 20E45, 20D05.

This is exactly the dual of Huppert's Conjecture on character degree sets of nonabelian simple groups, which was formulated by B. Huppert in the late 1990. If $\mathrm{cd}(G)$ denotes the character degree set of G, then we have:

Conjecture 2. (Huppert's Conjecture). *Let G be a finite group and let H be a nonabelian simple group. If $\mathrm{cd}(G) = \mathrm{cd}(H)$, then $G \cong H \times A$, where A is abelian.*

Note that Dual of Huppert's conjecture is an extension of Thompson's conjecture:

Conjecture 3. (Thompson's Conjecture). *Let G be a finite group with trivial center and let H be a nonabelian simple group. If $\mathrm{cs}(G) = \mathrm{cs}(H)$, then $G \cong H$.*

In [1], using the classification of finite simple groups, it is shown that $\mathrm{PSL}_n(q)$ satisfies Thompson's conjecture. In this article we want to improve this result and we will show that $\mathrm{PSL}_2(q)$ satisfies Dual of Huppert's Conjecture. At first we obtained the result by using the main theorem in [1]. However, we have chosen to avoid direct reference to the main theorem of [1], in order to keep our proofs independent on the classification of the finite simple groups. The proof is divided into two parts. First, we consider a finite group G such that $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_2(q))$, when q is even and to get the result in this case, we will use the classification of *F-groups*. By an F-group we mean a family of all finite groups G in which for any $x, y \in G - \mathbf{Z}(G)$, if $C_G(x) \leq C_G(y)$, then $C_G(x) = C_G(y)$. For a noncentral element $x \in G$, the centralizer $C_G(x)$ is said to be *free* if it is both maximal and minimal among all the proper centralizers of G. A group G is F-group, if and only if all of the centralizers of its non-central elements are free. The F-groups have been classified by Rebmann [11]. Secondly we study a group G such that $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_2(q))$, for odd number q. In this case we get the results by applying Baer and Suzuki's results on groups having a non-trivial partition. A set $\beta = \{H_1, H_2, ..., H_n\}$ of subgroups H_i $(i = 1, 2, ..., n)$ is said to be a *partition* of G if every element $x \in G$, $x \neq 1$, belongs to one and only one subgroup $H_i \in \beta$. If $n = 1$, the partition is said to be trivial.

Also we used the classification of *Zassenhaus group* of degree $q + 1$. In fact, a Zassenhaus group of degree $q + 1$ is a permutation group G of degree $q + 1$, in which the following hold:
(1) G is doubly transitive on $q + 1$ points.
(2) Any non-identity element has at most two fixed points.
(3) G has no regular normal subgroups.

Note that, if G acts on Ω and $K \leq G$ such that K acts on Ω transitively and also for every $\alpha \in \Omega$ the stabilizer K_α is trivial, then we say K is a regular subgroup of G.

For an integer n, we write $\pi(n)$ for the set of all prime divisors of n. We denote by $\pi(G)$, the set of all prime divisors of $|G|$. If p is a prime number and n is an integer, then we use the notation n_p for p-part of n, that is , $n_p = p^a$, where $p^a \mid n$ and $p^{a+1} \nmid n$. If π is a set of primes, by G_π we mean a Hall π-subgroup of G and in the particular case if p is a prime, then G_p denotes a Sylow p-subgroup of G. If $x \in G$, by index of x in G we mean the size of the conjugacy class containing x. All further unexplained notations are standard and are referred to [7], for example.

2 Main Results

The following result is the characterization of Zassenhaus groups of degree $q + 1$, which follows from [9, Theorem 11.6].

Theorem 2.1. *Suppose that G is a Zassenhaus group of degree $q + 1$ and order $(q + 1)qd$. Then $q = p^f$ is a prime-power, and the followings are the only possibilities.*
(a) $G = \mathrm{PGL}_2(p^f)$.
(b) p is odd and $G = \mathrm{PSL}_2(p^f)$.
(c) p is odd, f is even and G is a certain sharply triply transitive group of order $(q^2 - 1)q$ containing $\mathrm{PSL}_2(p^f)$ as its subgroup.

(d) $p = 2$ *and* G *is a Suzuki group* $Sz(q)$.

The following lemma describes the structure of a free centralizer which will be used frequently.

Lemma 2.2. ([5, Lemma 6]) *Let* G *be a group. If* $C_G(x)$ *is free, then either* $C_G(x)$ *is abelian or* $C_G(x) = U_p \times \mathbf{Z}(G)_{p'}$, *where* U_p *is a* p*-group for some prime* p.

Lemma 2.3. ([2]) *Let* G *be a finite group. If the set of* p*-regular conjugacy class sizes of* G *is exactly* $\{1, m\}$, *then* $m = p^a q^b$, *where* q *is a prime distinct from* p *and* $a \geq 0, b \geq 0$. *If* $b = 0$, *then* G *has an abelian* p*-complement. If* $b \neq 0$, *then* $G = PQ \times A$, *with* $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ *and* $A \leq \mathbf{Z}(G)$. *Furthermore, if* $a = 0$, *then* $G = P \times Q \times A$.

Lemma 2.4. *Let* G *be a finite group with order* np^α, *where* $(n, p) = 1$ *and* $\alpha > 0$. *Assume* P *and* R *are two Sylow* p*-subgroups of* G, *such that* $P \cap R$ *has index* p *in* P. *Then* $|G : N_G(P \cap R)| = n/t$, *where* $t \mid n$ *and* $t > p$.

Proof. It is easy to see that $RP \subseteq N_G(P \cap R)$. Hence $|N_G(P \cap R)| \geq |PR|$. Now the result is obvious. □

Lemma 2.5. *Let* $H \leq G$ *such that* $G = H\mathbf{Z}(G)$. *Then* $\mathrm{cs}(G) = \mathrm{cs}(H)$.

Proof. Since $G = HZ(G)$, this implies that $G = HC_G(r)$, for every $r \in G$. Therefore, for every $r \in G$, $|r^G| = |G : C_G(r)| = |HC_G(r) : C_G(r)| = |H : H \cap C_G(r)| = |H : C_H(r)| = |r^H|$. □

The following statement is taken from [8, Theorem A], which is the classification of F-groups.

Lemma 2.6. (Rebmann). *Let* G *be a nonabelian group. Then* G *is an* F*-group if and only if it is one of the following groups:*

(1) G *has a normal abelian subgroup of prime index.*

(2) $G/\mathbf{Z}(G)$ *is a Frobenius group with Frobenius kernel* $K/\mathbf{Z}(G)$ *and Frobenius complement* $L/\mathbf{Z}(G)$, *where* K *and* L *are abelian.*

(3) $G/\mathbf{Z}(G)$ *is a Frobenius group with Frobenius kernel* $K/\mathbf{Z}(G)$ *and Frobenius complement* $L/\mathbf{Z}(G)$, *such that* $K = P\mathbf{Z}(G)$, *where* P *is a normal Sylow* p*-subgroup of* G *for some prime* $p \in \pi(G)$, P *is an* F*-group,* $Z(P) = P \cap \mathbf{Z}(G)$ *and* $L = H\mathbf{Z}(G)$, *where* H *is an abelian* p'*-subgroup of* G.

(4) $G/\mathbf{Z}(G) \cong S_4$ *and if* $V/\mathbf{Z}(G)$ *is the Klein four group in* $G/\mathbf{Z}(G)$, *then* V *is nonabelian.*

(5) $G = P \times A$, *where* P *is a nonabelian* F*-group of prime power order and* A *is abelian.*

(6) $G/\mathbf{Z}(G) \cong \mathrm{PSL}_2(p^m)$ *or* $\mathrm{PGL}_2(p^m)$ *and* $G' \cong \mathrm{SL}_2(p^m)$, *where* p *is a prime and* $p^m > 3$.

(7) $G/\mathbf{Z}(G) \cong \mathrm{PSL}_2(9)$ *or* $\mathrm{PGL}_2(9)$ *and* G' *is isomorphic to the Schur cover of* $\mathrm{PSL}_2(9) \cong A_6$.

We note that except for the last two cases, G is solvable.

Lemma 2.7. ([3, 4]) *If a solvable group* G *has a non-trivial partition* β, *then one of the following conditions is satisfied:*

(i) *A component of* β *is selfnormalized and* G *is Frobenius.*

(ii) $G \cong S_4$ *and* β *consists of maximal cyclic subgroups of* G.

(iii) G *has a nilpotent normal subgroup* N, *which is a component of* β *such that* $|G : N| = p$ *and all* $x \in G \setminus N$ *has order* p.

(iv) G *is a* p*-group.*

Lemma 2.8. ([12]) *A finite non-solvable group* G *has a non-trivial partition* β *if and only if* G *is isomorphic to one of the following groups:*

(i) $G \cong Sz(q)$, *the Suzuki simple group.*

(ii) $G \cong \mathrm{PSL}_2(p^f)$ *or* $\mathrm{PGL}_2(p^f)$, *where* p *is a prime, and* $p^f \geq 4$.

Since $\mathrm{PSL}_2(2)$ is not simple, the following theorem covers all finite simple groups $\mathrm{PSL}_2(q)$, for even q.

Theorem 2.9. *Let G be a finite group and $q > 2$ even. If $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_2(q))$, then $G \cong \mathrm{PSL}_2(q) \times A$, where A is abelian.*

Proof. It is known that if $q = 2^f$, with $f \geq 2$, then

$$\mathrm{cs}(\mathrm{PSL}_2(q)) = \{1, q(q-1), q^2 - 1, q(q+1)\} = \{1, 2^f(2^f - 1), 2^{2f} - 1, 2^f(2^f + 1)\}.$$

Since there are no divisibilities among the nontrivial conjugacy class sizes of $\mathrm{PSL}_2(2^f)$, the centralizers of all non-trivial elements of G are free. Hence, G is an F-group. By using the classification of F-groups we have the following possibilities:

Case 1: G has a normal abelian subgroup H of prime index p.

If $H = \mathbf{Z}(G)$, then G is abelian since G/H is cyclic. Hence, $H - \mathbf{Z}(G)$ is nonempty. For $x \in H - \mathbf{Z}(G)$, we have $C_G(x) = H$ and then $|x^G| = p \notin \mathrm{cs}(G)$. So G is not in this class.

Case 2: $G/\mathbf{Z}(G)$ is a Frobenius group with Frobenius kernel $K/\mathbf{Z}(G)$ and Frobenius complement $L/\mathbf{Z}(G)$, where K and L are abelian.

Pick $x \in L - \mathbf{Z}(G)$, we have $C_G(x) = L$. Hence $|x^G| = [G : L] = |K|/|\mathbf{Z}(G)|$. Similarly, pick $y \in K - \mathbf{Z}(G)$ we have $|y^G| = |L|/|\mathbf{Z}(G)|$. So $\gcd(|x^G|, |y^G|) = 1$ which is not possible according to $\mathrm{cs}(G)$. Hence G is not in this class.

Case 3: $G/\mathbf{Z}(G)$ is a Frobenius group with Frobenius kernel $K/\mathbf{Z}(G)$ and Frobenius complement $L/\mathbf{Z}(G)$ such that $K = P\mathbf{Z}(G)$, where P is a normal Sylow p-subgroup of G for some prime $p \in \pi(G)$, P is an F-group, $\mathbf{Z}(P) = P \cap \mathbf{Z}(G)$ and $L = H\mathbf{Z}(G)$, where H is an abelian p'-subgroup of G.

It is clear that $G = KL$ and $K \cap L = \mathbf{Z}(G)$. Hence $|K| = |P||\mathbf{Z}(G)|/|P \cap \mathbf{Z}(G)|$ and $|G| = |KL| = |K||L|/|\mathbf{Z}(G)|$. Pick $x \in L - \mathbf{Z}(G)$ we have $C_G(x) = L$. Thus $|x^G| = |G|/|L| = |K|/|\mathbf{Z}(G)| = |P|/|P \cap \mathbf{Z}(G)|$ is a prime power which is not possible considering $\mathrm{cs}(G)$. So G is not in this class.

Case 4: $G/\mathbf{Z}(G) \cong S_4$ and if $V/\mathbf{Z}(G)$ is the Klein four group in $G/\mathbf{Z}(G)$, then V is nonabelian.

For each $x \in G - \mathbf{Z}(G)$ we have $\mathbf{Z}(G) \leq C_G(x)$ and $[G : C_G(x)] \mid [G : \mathbf{Z}(G)] = |S_4| = 2^3{\cdot}3$. This shows that all primes dividing conjugacy class sizes are either 2 or 3, which contradicts the fact that $2^f - 1, 2^f + 1$ and 2^f are pairwise coprime. Hence G is not in this class.

Case 5: $G = P \times A$, where P is a nonabelian F-group of prime power order and A is abelian.

It is clear that the conjugacy class sizes are all prime powers. Therefore, G is not in this class.

Case 6: $G/\mathbf{Z}(G) \cong \mathrm{PSL}_2(p^m)$ or $\mathrm{PGL}_2(p^m)$ and $G' \cong \mathrm{SL}_2(p^m)$, where p is a prime and $p^m > 3$.

If $p^m = 5$, then $\mathrm{cs}(G') = \{1, 12, 20, 30\}$ and $|G/\mathbf{Z}(G)| = 60$ or $120 = 2^3 \cdot 15$. Since $|x^G| = [G : C_G(x)] \mid [G : \mathbf{Z}(G)]$ for all $x \in G$, we have $2^f \mid 2^3$ and $2^f \pm 1 \mid 15$. Since $f \geq 2$, it implies $f = 2$. So $\mathrm{cs}(G) = \{1, 12, 15, 20\}$ which contains no multiples of $30 \in \mathrm{cs}(G')$. So G does not satisfy this case.

Suppose that $p^m \geq 7$ is odd and $p^m \equiv \nu \pmod 4$, where $\nu \in \{-1, 1\}$. We have $|\mathrm{SL}_2(p^m)| = |\mathrm{PGL}_2(p^m)| = p^m(p^{2m} - 1)$, and $\mathrm{cs}(\mathrm{SL}_2(p^m)) = \{1, (p^{2m} - 1)/2, p^m(p^m - 1), p^m(p^m + 1)\}$.

Since each $r \in \mathrm{cs}(G')$ is a divisor of some element in $\mathrm{cs}(G)$ and $p^m(p^m + \nu)$ is even, we conclude that $p^m(p^m + \nu)$ is a divisor of either $2^f(2^f - 1)$ or $2^f(2^f + 1)$. Since $p^m(p^m + \nu)/2$ is odd, we infer that $p^m(p^m + \nu)/2$ is a divisor of either $2^f - 1$ or $2^f + 1$. Thus $p^m(p^m - 1)/2 \leq p^m(p^m + \nu)/2 \leq 2^f + 1$. By $p^m \geq 7$, we have $2(p^m + 1) < p^m(p^m - 1)/2 - 2$. So $2(p^m + 1) < 2^f - 1$. Therefore, $p^m(p^{2m} - 1) < 2^{2f} - 1$. On the other hand, for each $x \in G$, we have $|x^G| = [G : C_G(x)] \mid [G : \mathbf{Z}(G)] = p^m(p^{2m} - 1)$. Hence $s \leq p^m(p^{2m} - 1)$ for all $s \in \mathrm{cs}(G)$, which is a contradiction, by the above argument. Thus G does not satisfy this case.

Now we suppose that p^m is even. So $G/\mathbf{Z}(G) \cong G' \cong \mathrm{PSL}_2(2^m)$, hence

$$\mathrm{cs}(G') = \{1, 2^m(2^m - 1), 2^{2m} - 1, 2^m(2^m + 1)\}$$

and $|G/\mathbf{Z}(G)| = 2^m(2^{2m} - 1)$. Since every $r \in \mathrm{cs}(G')$ is a divisor of some number in $\mathrm{cs}(G)$, we have $2^m \mid 2^f$. Since all $r \in \mathrm{cs}(G)$ are divisors of $|G/\mathbf{Z}(G)|$, we have $2^f \mid 2^m$. Thus $2^f = 2^m$ and so $m = f$.

It is clear that $G' \cap \mathbf{Z}(G) = \{1\}$ and $|G/G'| = |\mathbf{Z}(G)|$. Hence $G = G'\mathbf{Z}(G) \cong G' \times \mathbf{Z}(G)$, which is the result.

Case 7: $G/\mathbf{Z}(G) \cong \mathrm{PSL}_2(9)$ or $\mathrm{PGL}_2(9)$ and G' is isomorphic to a Schur cover of $\mathrm{PSL}_2(9) \cong A_6$.

We have $|\mathrm{PSL}_2(9)| = 360$, and $|\mathrm{PGL}_2(9)| = 720$. Using GAP, we have $90 \in \mathrm{cs}(G')$ for any Schur cover of A_6.

Since $720 = 2^4 \cdot 45$, by considering $\mathrm{cs}(G)$ and the fact that $|x^G| \mid [G : \mathbf{Z}(G)]$ for all $x \in G$, we obtain $2^f \mid 2^4$ and $2^f \pm 1 \mid 45$. By checking directly with $f = 2, 3, 4$, we have $f = 2$. So $\mathrm{cs}(G) = \{1, 12, 15, 20\}$ which contains no multiple of $90 \in \mathrm{cs}(G')$. Thus G does not satisfy this case. □

Note that $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$ and $\mathrm{PSL}_2(3) \cong A_4$ is not simple. The following theorem covers all finite simple groups $\mathrm{PSL}_2(q)$, for odd number q.

Theorem 2.10. *Let G be a group such that $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_2(q))$, where $q = p^f \geq 7$ and p is an odd prime. Then $G/\mathbf{Z}(G) \cong \mathrm{PSL}_2(q)$.*

The proof of Theorem 2.10 follows from a series of Lemmas and Remarks 2.11 - 2.18. In the following we assume that G satisfies the hypothesis of Theorem 2.10.

Remark 2.11. Since $q \equiv \nu \pmod 4$, where $\nu \in \{-1, +1\}$, we have

$$\mathrm{cs}(\mathrm{PSL}_2(q)) = \{1,\ (q^2-1)/2,\ q(q+\nu)/2,\ q(q-\nu),\ q(q+\nu)\}.$$

By a well-known result in [6] we may assume that $\pi(G) = \pi(q(q^2-1))$. Since $q \equiv \nu \pmod 4$, $(q+\nu)/2$ is odd and $(q-\nu)/2$ is even. So $(q+\nu)/2,\ q-\nu$ and q are pairwise coprime. Hence there is no prime $t \in \pi(G)$ such that $t \mid a$ for all $a \in \mathrm{cs}(G) - \{1\}$.

Lemma 2.12. *For each $t \in \pi(q(q+\nu)/2)$, G_t is abelian.*

Proof. First we claim for each $t \in \pi(q(q+\nu)/2)$, we have $\mathbf{Z}(G_t) - \mathbf{Z}(G)$ is nonempty. Let x be a noncentral element of G such that $G_t \leq C_G(x)$. Therefore $|x^G| \in \{(q^2-1)/2, q(q-\nu)\}$, and so $C_G(x)$ is free. By Lemma 2.2, $C_G(x)$ is either abelian or isomorphic to $U_s \times \mathbf{Z}(G)_{s'}$, where U_s is an s-group for some prime divisor s of $|G|$.

Suppose that $C_G(x)$ is not abelian. So $C_G(x) = U_s \times \mathbf{Z}(G)_{s'}$, where U_s is an s-group and $s \in \pi(G)$. Note that $G_t \leq C_G(x)$. If $s \neq t$, then $G_t \leq \mathbf{Z}(G)_{s'} \leq \mathbf{Z}(G)$ which is impossible, by the fact that t is a divisor of some conjugacy class size of G. Hence $s = t$ and $C_G(x) = G_t \times \mathbf{Z}(G)_{t'}$. So we can assume that x is a t-element and we conclude that $x \in \mathbf{Z}(G_t) - \mathbf{Z}(G)$.

Now we may assume $C_G(x)$ is abelian. Since $G_t \leq C_G(x)$, it is clear that G_t is abelian and so $\mathbf{Z}(G_t) = G_t$. If $G_t \leq \mathbf{Z}(G)$, then t does not divide any conjugacy class sizes, a contradiction. So our claim is proved.

Let $N = q(q^2-1)/2$. For each $t \in \pi(q(q+\nu)/2)$, we shall show that G_t is abelian by contradiction. We assume that G_t is not abelian. For every $y \in \mathbf{Z}(G_t) - \mathbf{Z}(G)$, we have $C_G(y) = G_t \times \mathbf{Z}(G)_{t'}$. Hence $C_G(\mathbf{Z}(G_t)) = \cap_{y \in \mathbf{Z}(G_t)} C_G(y) = G_t \times \mathbf{Z}(G)_{t'}$. So there is no non-central t'-element centralizing $\mathbf{Z}(G_t)$.

Since G_t is not abelian, there is $u \in G_t - \mathbf{Z}(G_t)$. So $\mathbf{Z}(G_t) \lneq C_G(u)_t$. By $\mathrm{cs}(G)$, we infer that $|G_t| = N_t |C_G(u)_t| > N_t |\mathbf{Z}(G_t)|$, where either $N_t = q$ or $N_t = ((q+\nu)/2)_t$.

If $t \in \pi((q+\nu)/2)$, then let $s = p$; otherwise if $t = p$, then choose $s \in \pi((q+\nu)/2)$. Pick $x \in \mathbf{Z}(G_s) - \mathbf{Z}(G)$. We have $|G : C_G(x)| \in \{q(q-\nu), (q^2-1)/2\}$. Without loss of generality we may assume $C_G(x)_t \lneq G_t$. Note that $|G_t : C_G(x)_t| = N_t$ and hence $|C_G(x)_t| > |\mathbf{Z}(G_t)|$, which implies that there is $y \in C_G(x)_t - \mathbf{Z}(G_t) \subset G_t$. But we know that $C_G(x)$ is free and so either $C_G(x)$ is abelian or $C_G(x) = G_s \times \mathbf{Z}(G)_{s'}$. By the fact that $|C_G(x)_t| > |\mathbf{Z}(G_t)|$, we deduce that $C_G(x)$ is abelain.

Note that $\mathbf{Z}(G_t) \lneq C_G(y)$. By the fact that $C_G(x)$ is abelian, we have $C_G(x) \leq C_G(y)$. Since $C_G(x)$ is free and $C_G(y) \lneq G$, we have $C_G(x) = C_G(y)$. Hence $\mathbf{Z}(G_t) \leq C_G(x)$ and so x centralizes $\mathbf{Z}(G_t)$. Since x is a t'-element, it contradicts the above argument. Thus, G_t is abelian. □

Remark 2.13. By Lemmas 2.12 and 2.2, for every $x \in G$ such that $|x^G| \in \{(q^2-1)/2, q(q-\nu)\}$, $C_G(x)$ is abelian.

Lemma 2.14. *Let $g \in G - \mathbf{Z}(G)$. The following hold.*

(i) If $|g^G| \in \{q(q+\nu), q(q+\nu)/2\}$, then $C_G(g)/\mathbf{Z}(G)$ is a $\pi(q-\nu)$-group.

(ii) If $|g^G| = q(q-\nu)$, then $C_G(g)/\mathbf{Z}(G)$ is a $\pi((q+\nu)/2)$-group.

(iii) If $|g^G| = (q^2-1)/2$, then $C_G(g)/\mathbf{Z}(G)$ is a p-group.

Proof. (i) Assume $g \in G$ such that $|g^G| \in \{q(q+\nu),\ q(q+\nu)/2\}$. Let $t \in \pi(q(q+\nu)/2)$. Write $g = g_t g_{t'} = g_{t'} g_t$, where g_t and $g_{t'}$ are t-part and t'-part of g, respectively (it means g_t is a t-element and $g_{t'}$ is a t'-element). We know $C_G(g) = C_G(g_t) \cap C_G(g_{t'})$. If $g_t \notin \mathbf{Z}(G)$, then $G_t \leq C_G(g_t)$ and $|g_t^G| = |G : C_G(g_t)|$ divides $|G : C_G(g)| = |g^G|$ which is impossible since by Lemma 2.12 we have $|g_t^G| \in \{(q^2-1)/2, q(q-\nu)\}$. Hence $g_t \in \mathbf{Z}(G)$. Now let $x \in C_G(g)$, be a t-element, for some $t \in \pi(q(q+\nu)/2)$. So $C_G(g_{t'}x) = C_G(x) \cap C_G(g_{t'})$. Hence $|g_{t'}^G| \mid |(xg_{t'})^G|$ and so $|(xg_{t'})^G| \in \{q(q+\nu),\ q(q+\nu)/2\}$. Therefore similar to the above discussion we conclude x is central. So $C_G(g)/\mathbf{Z}(G)$ is a $\pi(q-\nu)$-group. Thus part (i) is proved.

(ii) Let x be an element whose index is $q(q-\nu)$. By Remark 2.13, $C_G(x)$ is abelian. If there is a p-element $y \in C_G(x) - \mathbf{Z}(G)$, then $C_G(x) \leq C_G(y)$, and so $|y^G|$ divides $|x^G|$, which contradicts $|y^G| = (q^2-1)/2$.

Now we assume that there is a t-element $y \in C_G(x) - \mathbf{Z}(G)$, for some $t \in \pi(q-\nu)$. By Remark 2.13, $C_G(x)$ is abelian and free, it implies that $C_G(y) = C_G(x)$. From the set of conjugacy class sizes and part (i), it is clear that there exists a $\pi(q-\nu)$-element $z \in C_G(y)$ whose index is $q(q+\nu)/2$ and centralizes a G_s, for all $s \in \pi(q-\nu)$. Since $C_G(y)$ is abelian, we have $C_G(y) \leq C_G(z)$. So $|z^G| = |G : C_G(z)|$ divides $|G : C_G(y)| = |y^G| = q(q-\nu)$, a contradiction. Therefore $C_G(x)/\mathbf{Z}(G)$ is a $\pi((q+\nu)/2)$-group.

(iii) Let $x \in G$ have index $(q^2-1)/2$. If there is a t-element $y \in C_G(x) - \mathbf{Z}(G)$ for some $t \in \pi((q+\nu)/2)$, then by Lemma 2.12 $C_G(x) \leq C_G(y)$, and so $|y^G|$ divides $|x^G|$, which is impossible, since $|y^G| = q(q-\nu)$.

Now we assume that there is a t-element $y \in C_G(x) - \mathbf{Z}(G)$ for some $t \in \pi(q-\nu)$. Since $C_G(x)$ is free, it implies that $C_G(y) = C_G(x)$. Again by part (i), there exists a $\pi(q-\nu)$-element $z \in C_G(y)$ of index $q(q+\nu)/2$. Thus $|z^G| = |G : C_G(z)|$ divides $|G : C_G(y)| = |y^G| = (q^2-1)/2$, a contradiction. Thus $C_G(x)/\mathbf{Z}(G)$ is a p-group. □

In the following, we set $\bar{G} = G/\mathbf{Z}(G)$ and $\bar{x} = x\mathbf{Z}(G) \in \bar{G}$ for every $x \in G$.

Lemma 2.15. *The following hold.*

(i) $|\bar{G}| = q(q^2-1)/2$,

(ii) Let $\bar{x}$ and $\bar{y}$ are an s-element and a t-element in $\bar{G}$, respectively, when either $s \in \pi(q-\nu)$ and $t \in \pi((q+\nu)/2)$, $s = p$ and $t \in \pi(q-\nu)$, or $s = p$ and $t \in \pi((q+\nu)/2)$. Then $[\bar{x}, \bar{y}] \neq 1$.

(iii) G has Hall $\pi(q-\nu)$-subgroups and Hall $\pi((q+\nu)/2)$-subgroups.

Proof. (i) By Lemma 2.14, there exists $x \in G$ such that $|G : C_G(x)|_p = q$, and $|C_G(x)/\mathbf{Z}(G)|_p = 1$. Hence, $|G : \mathbf{Z}(G)|_p = q$. Similarly we conclude that $|G : \mathbf{Z}(G)|_t = ((q^2-1)/2)_t$, for every prime $t \in \pi((q^2-1)/2)$. Therefore, $|\bar{G}| = q(q^2-1)/2$.

(ii) Suppose that there are an s-element $\bar{x}$ and a t-element $\bar{y}$ in $\bar{G}$ such that $\bar{x}\bar{y} = \bar{y}\bar{x}$, where $s \in \pi(q-\nu)$, $t \in \pi((q+\nu)/2)$. Then we may assume x and y are an s-element and a t-element, respectively. We have $[x, y] \in \mathbf{Z}(G)$. If $o(x) = k$, then $[x^k, y] = [x, y^k] = 1$, so $x \in C_G(y^k) = C_G(y)$ since $\gcd(k, t) = 1$. This is a contradiction by Lemma 2.14 (ii). We apply the same argument for the pair (s, t) where either $s = p$ and $t \in \pi((q+\nu)/2)$, or $s = p$ and $t \in \pi(q-\nu)$, then we get another contradiction.

(iii) By Lemma 2.14 (i) and (ii), $\bar{G}$ has Hall $\pi(q-\nu)$-subgroups and Hall $\pi((q+\nu)/2)$-subgroups. Let $K/\mathbf{Z}(G) = C_G(y)/\mathbf{Z}(G)$ be a Hall $\pi(q-\nu)$-subgroup of $\bar{G}$, for some $y \in G$. So $K/(\mathbf{Z}(G)_{\pi(q-\nu)'})$ is a $\pi(q-\nu)$-group. By Schur-Zassenhaus Theorem, there exists $H \leq G$ such that $K = H \ltimes \mathbf{Z}(G)_{\pi(q-\nu)'} = H \times \mathbf{Z}(G)_{\pi(q-\nu)'}$ and H is a $\pi(q-\nu)$-group. It is easy to see that H is a Hall $\pi(q-\nu)$-subgroup of G. We can discuss similarly to deduce that G has Hall $\pi((q+\nu)/2)$-subgroups. □

Lemma 2.16. *Let $\pi(q-\nu) - \{2\} \neq \emptyset$. Assume that H is a Hall $\pi(q-\nu)$-subgroup of G. Then one of the following holds.*

(i) $H = Q \times A$, where Q is a Sylow 2-subgroup of H and A is a normal abelian 2-complement of H,

(ii) $H = Q \ltimes A$, where Q is a Sylow 2-subgroup of H and A is an abelian 2-complement of H. Furthermore, $|x^G| = q(q+\nu)$ for every $2'$-element $x \in H - \mathbf{Z}(G)$, and $C_G(x)$ is abelian for every $x \in H$ with $|x^G| = q(q+\nu)$.

Proof. Pick $y \in G$ such that $|y^G| = q(q+\nu)/2$. As we discussed in the proof of 2.15 (iii), $C_G(y) = H \times \mathbf{Z}(G)_{\pi(q-\nu)'}$ where H is the Hall $\pi(q-\nu)$-subgroup of G. By the maximality of $C_G(y)$, we may assume y is a t-element, for some prime t. Let $x \in C_G(y)$ be a t'-element. Note that there exists such an element, since $|\pi(q-\nu)| \geq 2$. Since $C_{C_G(y)}(x) = C_G(xy) = C_G(x) \cap C_G(y) \leq C_G(y)$ and from $\mathrm{cs}(G)$, the index of x in $C_G(y)$ is either 1 or 2. Since $H \trianglelefteq C_G(y)$, it follows that the index of every t'-elements of H is 1 or 2.

First we assume the indices of all t'-elements of H are 1. So H has a central t-complement. If $t \neq 2$, then $H = T \times A$, where T is a Sylow t-subgroup of G and A is an abelian 2-complement of H. This implies that $C_G(x)$ contains a Sylow 2-subgroup of G for all $x \in H$, and thus, $|x^G| = q(q+\nu)/2$, which contradicts Lemma 2.14. So $t = 2$ and we have $H = Q \times A$, where Q is a Sylow 2-subgroup of H and A is an abelian 2-complement of H, as we claimed in (i).

Secondly, we assume the set of indices of t'-elements of H equals $\{1, 2\}$. By Lemma 2.3, we consider the following two cases of t:

(a) If $t \neq 2$, then $H = T \times Q \times A'$, where $T \in \mathrm{Syl}_t(H)$, $Q \in \mathrm{Syl}_2(H)$ and $A' \leq \mathbf{Z}(H)$. From $\mathrm{cs}(G)$, every $2'$-element $z \in H$ has $|z^G| = q(q+\nu)/2 = |y^G|$. Thus, there must exist a 2-element x such that $|C_G(x)| = |C_G(y)|/2$. For all $2'$-elements $z \in C_G(x)$, by the minimality of $C_G(x)$ we have $C_G(xz) = C_G(x) \cap C_G(z) = C_G(x) \leq C_G(z)$, which implies $z \in \mathbf{Z}(C_G(x))$. So by setting $A = T \times A'$, we have $H = Q \times A$ where A is abelian, as we claimed in (i).

(b) If $t = 2$, by Lemma 2.3, H has abelian 2-complements. Here we may assume that all non-central $2'$-elements $x \in H$ have $|x^G| = q(q+\nu)$, since if there exists a $2'$-element x such that $|C_G(x)| = |C_G(y)|$, then similar to case (a), we deduce (i) holds.

First assume that $x \in H$ is a $2'$-elements such that $|x^G| = q(q+\nu)$. Then by the minimality of $C_G(x)$, we have $C_G(x)_2 \leq \mathbf{Z}(C_G(x))$. Since H has abelian 2-complements, $C_G(x)$ is abelian. Now let x be a 2-element such that $|x^G| = q(q+\nu)$. Let $z \in C_G(x)$ be a non-central $2'$-element. Then $C_G(x) \leq C_G(z)$ and so $C_G(x) = C_G(z)$, by the fact that $|C_G(z)| = |C_G(y)|/2 = |C_G(x)|$, for every non-central $2'$-element z. Since $C_G(z)$ is abelian, $C_G(x)$ is also abelian. Hence for every x with $|x^G| = q(q+\nu)$, $C_G(x)$ is abelian, as we claimed in (ii).

Let $|x^G| = q(q+\nu)$, for some $x \in G$. Then $C_G(x)_{\pi(q-\nu)} \leq H$, for some Hall $\pi(q-\nu)$-subgroup H and $|H : C_G(x)_{\pi(q-\nu)}| = 2$. Thus $C_G(x)_{\pi(q-\nu)} \trianglelefteq H$. We also know that $C_G(x)_{\pi(q-\nu)}$ is abelian and so $C_G(x)_{\pi(q-\nu)}$ has a normal 2-complement. Hence H has a normal 2-complement as well. Therefore $H = Q \ltimes A$, where A is the 2-complement of H and Q is a Sylow 2-subgroup of H and so (ii) holds. □

Lemma 2.17. *Let $\pi(q-\nu) - \{2\} \neq \emptyset$. Then $\bar{G} \cong \mathrm{PSL}(2, q)$.*

Proof. We shall show that $\bar{G}$ has a non-trivial partition. By Lemma 2.16, we have the following cases:

First, assume $H = Q \times A$, where H is a Hall $\pi(q-\nu)$-subgroup of G, Q is a Sylow 2-subgroup of G, and A is an abelian 2-complement of H. So by well-known Wielandt's Theorem (see [10, 9.1.10]), all Hall $\pi(q-\nu)$-subgroups of G are conjugate.

Let $\beta = \{C_G(x)/\mathbf{Z}(G) : x$ is a non-central element of G such that $C_G(x)$ is maximal in the lattice of centralizers of $G\}$. Since the elements in β are either Hall $\pi(q-\nu)$-subgroups, Hall $\pi(q+\nu)/2$-subgroups or Sylow p-subgroups of $\bar{G}$, we can see that union of elements of β is a cover for $\bar{G}$. Let $z \in C_G(x) \cap C_G(y) - \mathbf{Z}(G)$. By Lemma 2.14 and the maximality of $C_G(x), C_G(y)$, we have $|x^G| = |y^G|$. First, assume $|x^G| = |y^G| \in \{(q^2-1)/2, q(q-\nu)\}$. Since $C_G(x)$ is abelian by Remark 2.13, we have $C_G(x) \leq C_G(z)$ and also by the maximality of $C_G(x)$, we conclude that $C_G(x) = C_G(z)$. Similarly we have $C_G(y) = C_G(z)$ and so $C_G(x) = C_G(y)$.

Now let $|x^G| = |y^G| = q(q+\nu)/2$. As we discussed in the proof of Lemma 2.15 (iii) we may assume $C_G(x) = H \times \mathbf{Z}(G)_{\pi(q-\nu)'}$ and hence $C_G(y) = H^g \times \mathbf{Z}(G)_{\pi(q-\nu)'}$, for some $g \in G$.

Since A and A^g are central in $C_G(x)$ and $C_G(y)$, respectively, we have $C_G(z)$ contains A and A^g as its subgroups. Note that, considering the structure of $C_G(x)$, we may assume z is a non-central $\pi(q-\nu)$-element. So $|z^G| \in \{q(q+\nu), q(q+\nu)/2\}$, which implies that $C_G(z)_{\pi(q-\nu)} \leq H^k$, for some $k \in G$, by using Lemma 2.14. Thus A is normal in $C_G(z)$, which implies that $A = A^g$. Since $\pi(q-\nu) - \{2\} \neq \emptyset$, we have $\langle Q, Q^g \rangle \leq C_G(A) = C_G(x)$. By the fact that Q is normal in $C_G(x)$, we have $Q = Q^g$. Thus $C_G(x) = C_G(y) = H \times \mathbf{Z}(G)_{\pi(q-\nu)'}$. Therefore β is a partition for $\bar{G}$.

Secondly, assume $H = Q \ltimes A$, where Q is a Sylow 2-subgroup of H and A is a normal abelian 2-complement of H, $C_G(x)$ is abelian for every $x \in G$ with $|x^G| = q(q+\nu)$, and $|y^G| = q(q+\nu)$ for every $2'$-element $y \in H - \mathbf{Z}(G)$.

Let $\beta = \{C_G(x)/\mathbf{Z}(G) : x$ is a non-central element of G such that $C_G(x)$ is minimal in the lattice of centralizers of $G\}$. We claim that union of elements of β is a cover for $\bar{G}$. It is obvious that all of the elements of $\bar{G}$, beside 2-elements whose indices are $q(q+\nu)/2$, appear in the components of β. Assume x is a 2-element whose index in G is $q(q+\nu)/2$. Then $H \leq C_G(x)$, for some Hall $\pi(q-\nu)$-subgroup H of G. Thus $x \in \mathbf{Z}(H)$. To fulfil the claim that the union of components of β covers $\bar{G}$, it suffices to find $y \in H$ such that $|C_G(y)| = |C_G(x)|/2$, then $x \in C_G(y)$.

Let $z \in C_G(x) \cap C_G(y) - \mathbf{Z}(G)$. We know that $|x^G| = |y^G|$, by Lemma 2.14. If $|x^G| = |y^G| \in \{(q^2-1)/2, q(q-\nu)\}$, then similarly to the first case we get $C_G(x) = C_G(y)$. If $|x^G| = |y^G| = q(q+\nu)$, then, by the fact that $C_G(x)$ and $C_G(y)$ are abelian, we have $C_G(x) \leq C_G(z)$ and $C_G(y) \leq C_G(z)$. Thus either $C_G(x) = C_G(z) = C_G(y)$, or $C_G(z) = H \times \mathbf{Z}(G)_{\pi(q-\nu)'}$, for some Hall $\pi(q-\nu)$-subgroup H of G. The former case is what we wanted to prove, so we may assume the later case holds. Since H has a unique 2-complement, we deduce that $C_G(x)$, $C_G(y)$ and $C_G(z)$ have the same 2-complement. Let $C_G(x) = A \times Q_1 \times \mathbf{Z}(G)_{\pi(q-\nu)'}$ and $C_G(y) = A \times Q_2 \times \mathbf{Z}(G)_{\pi(q-\nu)'}$, where $A \times \mathbf{Z}(G)_{\pi(q-\nu)'}$ is the 2-complement of $C_G(x)$ and $C_G(y)$, and Q_1 and Q_2 are Sylow 2-subgroups of $C_G(x)$ and $C_G(y)$, respectively. For $u \in A - \mathbf{Z}(G)$ we have $|C_G(u)| = |C_G(x)| = |C_G(y)|$. Also $C_G(x) \leq C_G(u)$ and $C_G(y) \leq C_G(u)$. Hence $C_G(u) = C_G(x) = C_G(y)$. Therefore β is a partition for $\bar{G}$.

So by Lemmas 2.7 and 2.8, one of the following cases occurs:

Case 1. Similar to the argument we had in Theorem 2.9, we have $\bar{G}$ is not a p-group and also it is not isomorphic to S_4.

Case 2. Let $\bar{G}$ has a normal subgroup, say N, such that $|\bar{G} : N| = r$, where r is a prime number and N is nilpotent.

In this case N is one of the components of β and so $|N| \in \{q, (q-\nu), (q-\nu)/2, (q+\nu)/2\}$, which is not possible since $|\bar{G}/N|$ is prime.

Case 3. $\bar{G}$ is a Frobenious group.

Since the Frobenius kernel of $\bar{G}$ is nilpotent, we have K, the Frobenius kernel of $\bar{G}$, is either a Sylow p-subgroup of $\bar{G}$, a Hall $\pi(q-\nu)$-subgroup of $\bar{G}$, or a Hall $\pi((q+\nu)/2)$-subgroup of $\bar{G}$, by Lemma 2.15 (ii). If H is a Frobenius complement of $\bar{G}$, then $(|K|, |H|) \in \{(q-\nu, q(q+\nu)/2), ((q+\nu)/2, q(q-\nu)), (q, (q^2-1)/2)\}$, which is contradicting $|H| \mid |K| - 1$.

Case 4. $\bar{G}$ is isomorphic to $Sz(2^h)$, for some odd integer $h \geq 3$.

Then $|\bar{G}| = |Sz(2^h)| = 2^{2h}(2^{2h}+1)(2^h-1)$. It is well-known that 2-elements in $Sz(2^h)$ does not commute with any t-elements, for prime t different from 2. Hence $q - \nu = 2^{2h}$, which contradicts the hypothesis of this Lemma.

Case 6. $\bar{G}$ is isomorphic to $\mathrm{PGL}_2(r^h)$, where $r^h \geq 4$.

Since $|\mathrm{cs}(\mathrm{PGL}_2(r^h))| = 6$, when r^h is odd, we may assume $r = 2$. Hence $2^h \mid q-\nu$ and also $q \mid 2^h \pm 1$. Hence $2^h - 1 \leq q \leq 2^h + 1$ and so $q = 2^h \pm 1$. But we know that $(q+\nu)/2$ is odd and $\pi(q-\nu) \neq \{2\}$, a contradiction.

Case 7. $\bar{G}$ is isomorphic to $\mathrm{PSL}_2(r^h)$, where $r^h \geq 4$. If r is even, then, by the same discussion as we had in Case 6, we produce a contradiction. So we may assume r is odd. If $r \nmid q$, then $r^h \mid (q \pm 1)$ and $q \mid (r^h \pm 1)$. Thus $r^h - 1 \leq q \leq r^h + 1$ and so $q = r^h \pm 1$, a contradiction. Therefore $r^h = q$ and hence $\bar{G} \cong \mathrm{PSL}_2(q)$. □

Lemma 2.18. *Let $\pi(q-\nu) = \{2\}$. Then $\bar{G} \cong \mathrm{PSL}_2(q)$.*

Proof. Let $q - \nu = 2^\alpha$, for some integer α. Then either $q = p$ is prime, or $q = 9$. First, assume $q = p$ is prime. We show that $\bar{G}$ is a Zassenhaus group of degree $p+1$. We know that $|\bar{G}| = p(p^2-1)/2$.

Now we claim that either $n_p(\bar{G}) = p+1$ or $n_p(\bar{G}) = 1$. Let $n_p(\bar{G}) = mt = kp+1$, where m, t and k are integers such that $m \mid (p+1)$ and $t \mid (p-1)/2$. If either $m = p+1$ or $m = 1$, then $t = 1$, since $t \equiv 1 \pmod{p}$. So we may assume $1 < m \leq (p+1)/2$. Then we can write $t = (p-1)/2m'$, where $t \geq 2$ and hence $2m' \leq (p-1)/2$, since otherwise $1 < n_p(\bar{G}) < p+1$ and this is impossible. Considering $m(p-1)/2m' = kp+1$, we have $p \mid 2m' + m$. Using the fact $2m' \leq (p-1)/2$ and $m \leq (p+1)/2$, we conclude that $2m' = (p-1)/2$ and $m = (p+1)/2$. Thus $kp+1 = p+1$ as we claimed.

If $n_p(\bar{G}) = 1$, then $\bar{G} = H \ltimes P$, where $P \in \mathrm{Syl}_p(\bar{G})$ and H is a p-complement of $\bar{G}$. We know that $C_H(x) = 1$ for every $x \in P$, by Lemma 2.15 (ii). Hence $\bar{G}$ is a Frobenius group, which contradicts $(p^2-1)/2 = |H| \mid p-1$. So $n_p(\bar{G}) = p+1$.

Now, let $\Omega = \mathrm{Syl}_p(\bar{G})$. Obviously $\bar{G}$ acts on Ω transitively and the stabilizer $\bar{G}_P = N_{\bar{G}}(P)$, for every $P \in \Omega$. Hence $N_{\bar{G}}(P) = H \ltimes P$, where H is a subgroup of $\bar{G}$ whose order is $(p-1)/2$. Since $P \cap N_{\bar{G}}(P_0) = P \cap P_0 = 1$, for every $P_0 \in \Omega - \{P\}$, we infer that P acts on $\Omega - \{P\}$ transitively. So the stabilizer of P, $\bar{G}_P$, acts on $\Omega - \{P\}$ transitively, which implies that $\bar{G}$ is doubly transitive. It means $|\bar{G}_P : \bar{G}_{P_0} \cap \bar{G}_P| = p$ for every $P \neq P_0 \in \Omega$.

So we can say that for every $P \neq P_0 \in \Omega$, $\bar{G}_P \cap \bar{G}_{P_0}$ is a p-complement of $N_{\bar{G}}(P)$ and so we may assume $\bar{G}_P \cap \bar{G}_{P_0} = H$, for some p-complement H of $N_{\bar{G}}(P)$. Note that $N_{\bar{G}}(P)$ is a Frobenius group, by Lemma 2.15 (ii), and so $N_{\bar{G}}(P) = (\bigcup_{g\in P} H^g) \bigcup P$. Hence every p'-element in $N_{\bar{G}}(P)$ fixes at least one Sylow p-subgroup in $\Omega - \{P\}$. Thus, if we consider the action of $\bar{G}_P$ on $\Omega - \{P\}$, then we have $|\mathrm{Fix}(h)| \geq 1$, for every p'-element of $N_{\bar{G}}(P)$. As we discussed, non-identity elements in P does not fix any element in $\Omega - \{P\}$. Now considering the following equation,

$$|N_{\bar{G}}(P)| = p + \sum_{h \in N_{\bar{G}}(P) - P} |\mathrm{Fix}(h)|,$$

we will have $|\mathrm{Fix}(h)| = 1$. Therefore the elements in $\bar{G}$ fixes at most 2 points in Ω.

Let $K \trianglelefteq \bar{G}$ such that K acts on Ω transitively and also $K_P = 1$, for every $P \in \Omega$. Since the action of K on Ω is transitive, then $|K : K_P| = |\Omega| = p+1$, for every $P \in \Omega$. Therefore $|K| = p+1$. Now consider $P \ltimes K$. By Lemma 2.15 (ii), $C_P(k) = 1$, for every $k \in K$. Hence $P \ltimes K$ is a Frobenius group. This implies that K is nilpotent and so K is a subgroup of either a Hall $\pi(q-\nu)$-subgroup, or a Hall $\pi((q+\nu)/2)$-subgroup. So we conclude that $|K| = p+1 = q-\nu = 2^\alpha$. Now consider $\bar{G} = H \ltimes K$, where H is a 2-complement of $\bar{G}$. Again we can see that $\bar{G}$ is a Frobenius group, which is contradicting $|H| \nmid |K| - 1 = p$. Hence $\bar{G}$ does not have any regular normal subgroup. Therefore our claim is proved and $\bar{G}$ is a Zassenhaus group of degree $p+1$. Now using Theorem 2.1 and from Lemma 2.15 (i), we have $\bar{G} = \mathrm{PSL}_2(p)$, as wanted.

Secondly, let $q = 9$ and so $|\bar{G}| = 9.5.8$. Thus either $n_3(\bar{G}) = 1$, $n_3(\bar{G}) = 4$, $n_3(\bar{G}) = 40$ or $n_3(\bar{G}) = 10$. Similar to the first case we conclude that $n_3(\bar{G}) \neq 1$. Let $n_3(\bar{G}) = 4$. We know that $\bar{G}$ acts transitively on $\mathrm{Syl}_3(\bar{G})$. So for $P \in \mathrm{Syl}_3(\bar{G})$, $\bar{G}_P = N_{\bar{G}}(P) = H \ltimes P$, where $|H| = 10$. Moreover $N_{\bar{G}}(P)$ is a Frobenius group, which contradicts $10 \nmid 9-1$. If $n_3(\bar{G}) = 40$, then $C_{\bar{G}}(P) = N_{\bar{G}}(P)$, for $P \in \mathrm{Syl}_3(\bar{G})$.

Therefore $\bar{G}$ is 3-nilpotent and so $\bar{G} = P \ltimes K$, where K is the 3-complement of $\bar{G}$. As we discussed before $\bar{G}$ is a Frobenius group and hence K is nilpotent, a contradiction. Hence $n_3(\bar{G}) = 10$.

We claim that every two Sylow 3-subgroups have a trivial intersection. On the contrary, assume there exist $P,\ R \in \mathrm{Syl}_3(\bar{G})$, such that $S = (P \cap R)$ has index 3 in P. By Lemma 2.4, we have $|N_{\bar{G}}(S)| = 9t$, where $t \mid 40$ and $t \geq 4$. Considering Lemma 2.15 (ii), we have $|C_{\bar{G}}(S)| = 9$. By using Normalizer-Centralizer Theorem we have $t \leq 2$, a contradiction. Therefore our claim is proved.

Now using the fact that every two Sylow 3-subgroups have a trivial intersection, similar to the first case, we obtain G is a Zassenhaus group of degree 10 and so we have $G \cong \mathrm{PSL}_2(9)$, which is our desired result. □

The proof of Theorem 2.10 is an immediate consequence of Lemmas 2.17 and 2.18.

Theorem 2.19. *Let G be a finite group. If $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_2(q))$, with $q \geq 7$ odd, then $G \cong \mathrm{PSL}_2(q) \times A$, where A is an abelian group.*

Proof. By Theorem 2.10, it is enough to prove that either $\mathbf{Z}(G) = 1$ or $\mathbf{Z}(G)$ is a direct factor of G. We argue by minimal counterexample. So we assume G is a group with minimal order such that $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_2(q))$ and $\mathbf{Z}(G)$ is not trivial and it is not a direct factor of G.

By Theorem 2.10 we have $G'\mathbf{Z}(G) = G$ and $G'/(\mathbf{Z}(G) \cap G') \cong \mathrm{PSL}_2(q)$. Note that $G' \cap \mathbf{Z}(G) = \mathbf{Z}(G')$, since otherwise if $x \in \mathbf{Z}(G') - \mathbf{Z}(G)$, then $x(G' \cap \mathbf{Z}(G)) \in \mathbf{Z}(G'/(G' \cap \mathbf{Z}(G)))$, a contradiction. Now by Lemma 2.5, we conclude that $\mathrm{cs}(G) = \mathrm{cs}(G')$.

We claim that $G = G'$. Assume, on the contrary $G' < G$. But since G is a minimal counterexample such that its center is not trivial and is not a direct factor of G and $\mathrm{cs}(G) = \mathrm{cs}(\mathrm{PSL}_2(q))$, either $\mathbf{Z}(G') = \mathbf{Z}(G) \cap G' = 1$ or $G' = T \times \mathbf{Z}(G')$ and $T \cong \mathrm{PSL}_2(q)$.

Assume the former case occurs. Then $G \cong G' \times \mathbf{Z}(G) \cong \mathrm{PSL}_2(q) \times \mathbf{Z}(G)$, which is not possible by our assumption on G. So we may assume the latter case holds. Therefore $G = G'\mathbf{Z}(G) \cong (T \times \mathbf{Z}(G'))\mathbf{Z}(G) \cong T \times \mathbf{Z}(G)$ which again contradicts our assumption on G. Hence $G' = G$.

Now since G is perfect and $G/\mathbf{Z}(G)$ is simple, G is a quasi-simple group and so $\mathbf{Z}(G) \leq M(\mathrm{PSL}_2(q))$, where $M(\mathrm{PSL}_2(q))$ is the Schur multiplier of finite simple group $\mathrm{PSL}_2(q)$. If $q \neq 9$, then $|M(\mathrm{PSL}_2(q))| = 2$ and since $\mathbf{Z}(G)$ is not trivial, we have $\mathbf{Z}(G) = M(\mathrm{PSL}_2(q))$. Hence G is isomorphic to the unique Schur cover of $\mathrm{PSL}_2(q)$ which is $G \cong \mathrm{SL}(2, q)$. By considering the set of conjugacy class sizes of $\mathrm{SL}(2, q)$, we get a contradiction.

Now if $q = 9$, then G is a quotient of the Schur representation of $\mathrm{PSL}_2(9)$. In fact, if $6 \cdot A_6$ denotes the Schur representation of $\mathrm{PSL}_2(9)$, then $G \cong 6 \cdot A_6$, $3 \cdot A_6$ or $\mathrm{SL}_2(9)$. By checking the conjugacy class sizes of these groups, we produce a contradiction. □

Acknowledgements

The authors are grateful to the referees for their comments and suggestions.

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