RANK 3 PERMUTATION CHARACTERS AND MAXIMAL SUBGROUPS

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Abstract. Let G be a transitive permutation group acting on a finite set $\mathfrak{E}$ and let P be the stabilizer in G of a point in $\mathfrak{E}$. We say that G is primitive rank 3 on $\mathfrak{E}$ if P is maximal in G and P has exactly three orbits on $\mathfrak{E}$. For any subgroup H of G, we denote by 1_H^G the permutation character (or permutation module) over $\mathbb{C}$ of G on the cosets G/H. Let H and K be subgroups of G. We say $1_H^G \leqslant 1_K^G$ if $1_K^G - 1_H^G$ is a character of G. Also a finite group G is called nearly simple primitive rank 3 on $\mathfrak{E}$ if there exists a quasi-simple group L such that $L/Z(L) \trianglelefteq G/Z(L) \leqslant Aut(L/Z(L))$ and G acts as a primitive rank 3 permutation group on the cosets of some subgroup of L. In this paper we classify all maximal subgroups M of a nearly simple primitive rank 3 group G of type $L = \Omega_{2m+1}(3), m \geqslant 3$, acting on an L-orbit $\mathfrak{E}$ of non-singular points of the natural module for L such that $1_P^G \leqslant 1_M^G$, where P is the stabilizer of a non-singular point in $\mathfrak{E}$. This result has an application to the study of minimal genera of algebraic curves which admit group actions.

1. Introduction

Let G be a transitive permutation group acting on a finite set $\mathfrak{E}$ and let P be the stabilizer of a point in $\mathfrak{E}$. We say that G is *primitive* on $\mathfrak{E}$ if and only if P is maximal in G. We define the *rank* of G on $\mathfrak{E}$ to be the number of P-orbits on $\mathfrak{E}$. For any subgroup H of G, we denote by 1_H^G the permutation character of G on the cosets G/H over $\mathbb{C}$. We also use the same notation 1_H^G for the permutation module. Let H, K be subgroups of G. Consider the permutation characters 1_H^G and 1_K^G, we say $1_H^G \leqslant 1_K^G$ if $1_K^G - 1_H^G$ is a character of G. In terms of permutation modules, $1_H^G \leqslant 1_K^G$ if and only if 1_H^G is isomorphic to a submodule of 1_K^G. A finite group L is said to be *quasi-simple* if L is perfect and $L/Z(L)$ is simple. A finite group G is called *nearly simple of type* L if $L \trianglelefteq G$ and $L/Z(L) \leqslant G/Z(L) \leqslant Aut(L/Z(L))$ for some quasi-simple group L. Moreover a finite group G is called *almost simple of type* L or *almost simple with socle* L if $L \trianglelefteq G \leqslant Aut(L)$ for some finite simple group L. It follows from definitions that if G is nearly simple of type L then $G/Z(L)$ is almost simple of type $L/Z(L)$. Assume that $m \geqslant 3$ is an integer. Let L be one of the following quasi-simple groups $\Omega_{2m+1}(3), \Omega_{2m}^{\varepsilon}(3), \Omega_{2m}^{\varepsilon}(2)$ or $SU_m(2)$. Let G be a nearly simple group of type L such that G acts on the L-orbit $\mathfrak{E}(V)$ of non-singular points in the natural module V for L and let P be the stabilizer of a non-singular point in $\mathfrak{E}(V)$. Then G is a primitive rank 3 group on $\mathfrak{E}(V)$. In this situation, we say that G is a *nearly simple primitive rank* 3 *group of type* L. In this paper, we shall classify all maximal subgroups M of a nearly simple primitive rank 3 group G of

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TABLE 1. $M \in \mathcal{C}$

L	type of M	conditions	orbits	Ref
$\Omega_{2m+1}(3)$	$O_1(3) \perp O_{2m}^{\varepsilon}(3)$			5.3
	P_α	$1 \leqslant \alpha \leqslant m$	$\leqslant 2$	5.6
	$O_1(3) \wr S_{2m+1}$	$(2m+1, \xi, r) = (5, \pm, t)$	2	5.10
		$(7, +, t)$	2	
	$O_\alpha(3^3)$	$2m+1 = 3\alpha$	3	5.13

TABLE 2. $M \in \mathcal{S}$

L	socle of M	modules	orbits	Ref
$\Omega_7(3)$	A_9	$\lambda = (8,1)$	$\leqslant 2$	5.19
$\Omega_7(3)$	$PSp_6(2)$		2	5.20
$\Omega_{13}(3)$	$PSp_6(3)$	λ_2	$\leqslant 2$	5.23
$\Omega_7(3)$	$G_2(3)$	λ_1, λ_2	1	5.25
$\Omega_{25}(3)$	$F_4(3)$	λ_4	$\leqslant 2$	

TABLE 3. Possible exceptions

L	socle of M	modules	Ref
$\Omega_{41}(3)$	$S_8(3)$	λ_1	5.23
$\Omega_{77}(3)$	$E_6(3)$	adjoint module	5.25
$\Omega_{133}(3)$	$E_7(3)$	adjoint module	

type $\Omega_{2m+1}(3)$ such that $1_P^G \leqslant 1_M^G$. The remaining types of nearly simple primitive rank 3 groups of type L have been considered in [33].

Theorem 1.1. *Let $L = \Omega_{2m+1}(3)$, with $m \geqslant 3$, and let G be a nearly simple primitive rank 3 group of type L. Let P be the stabilizer of a non-singular point in V. Let M be any maximal subgroup of G. Then $1_P^G \leqslant 1_M^G$ unless the pairs (L, M) appear in Tables 1-3.*

Remark 1.2. *For all pairs (L, M) in Tables 1 and 2, 1_P^G is not contained in 1_M^G. The pairs in Table 3 are the cases that we have not determined whether or not there is a containment.*

The main motivation for this work comes from algebraic curves which admit group actions. From Riemann's Existence Theorem, we know that for every finite group there are infinitely many Riemann surfaces with automorphism group G. We would like to identify G-curves of smallest possible genus. Let X be a compact Riemann surface with $G \cong Aut(X)$. Let ℓ be a prime such that $(\ell, |G|) = 1$. Let J_X be the Jacobian of X and let $J_X[\ell]$ be the ℓ-torsion points of J_X. It is well known that $J_X[\ell]$ is a characteristic ℓ module and the genus of the quotient curve X/P can be computed using the relation $2g(X/P) = dim(J_X[\ell]^P)$, where $P \leqslant G$ and $J_X[\ell]^P$ is the fixed point space of $J_X[\ell]$ under P (see [9, Lemma 8.1]). Now combining the latter fact with the Frobenius reciprocity, we deduce that if P, M are proper subgroups of G such that $1_P^G \leqslant 1_M^G$, then $g(X/P) \leqslant g(X/M)$. For a proof of this result see [9, Corollary 8.2]. Theorem 1.1 with the discussion above now yield the following.

Corollary 1.3. *Assume the assumption and notation of Theorem 1.1. Let X be a compact Riemann surface with $G \leqslant Aut(X)$, and inertia groups $\langle g_1\rangle, \ldots, \langle g_r\rangle$ over X/G. Then $g(X/P) \leqslant g(X/M)$ for any maximal subgroup M of G which does not appear in Tables 1-3.*

If G is doubly transitive on a non-empty set $\mathfrak{E}$ with point stabilizer P, then either $1_P^G \leqslant 1_M^G$ or $G = PM$ for any maximal subgroup M of G. The maximal factorization of almost simple groups was classified completely by M. Liebeck, C. Praeger and J. Saxl in [25], so for almost simple groups, we can tell exactly for which M we have the containment $1_P^G \leqslant 1_M^G$. In case of rank 3, M. Aschbacher, R. Guralnick and K. Magaard [1] have a criterion in terms of the Higman rank 3 parameters. In that paper, they consider the case when G is a nearly simple classical group acting on the set of singular points on its natural module. Also a partial result of this case has been dealt with by D. Frohardt, the second and the third authors above in [9]. In the case when G is a nearly simple primitive rank 3 group of sporadic type, the containment of the permutation characters of G is completely determined since all the permutation characters of maximal subgroups of G are stored in [10], except $HS:2, Fi_{22}:2$ and $Fi'_{24}:2$.

We now describe our strategy. Let L be a finite simple classical group of degree $d \geqslant 2$, defined over a finite field $\mathbb{F}_q, q$ a prime power, and let V be the natural module for L. Assume that G is an almost simple group with simple socle L. We have a powerful theorem on the subgroup structure of G by M. Aschbacher. The theorem says that if M is a subgroup of G then M belongs to a collection $\mathcal{C}(G)$ of geometric subgroups of G or $M \in \mathcal{S}(G)$, that is, M is an almost simple group and the full covering group of the socle of M acts absolutely irreducible on the natural module V for G and cannot be realized over any proper subfield. Thus if M is a maximal subgroup of G then either $M \in \mathcal{C}(G)$ or $M \in \mathcal{S}(G)$. The subgroup structure and the maximality among members of $\mathcal{C}(G)$ have been determined by P. Kleidman and M. Liebeck in [21] when the degree is at least 13. For this case, using the geometrical properties of the groups, we can solve the problem completely. When M is not a geometric subgroup, that is, $M \in \mathcal{S}(G)$, the problem is much more complicated as we still do not know which members of $\mathcal{S}(G)$ are maximal. Now assume that $M \in \mathcal{S}(G)$. Denote by S the socle of M. So S is a non-abelian finite simple group. According to the Classification of Finite Simple Groups, S is an alternating group of degree at least 5, a finite group of Lie type or one of the 26 sporadic groups. By way of contradiction, we assume that $1_P^G \nleqslant 1_M^G$. From this assumption, we will get an upper bound for the dimension of V in terms of the size of the automorphism group of S. From the definition of members in $\mathcal{S}(G)$, the full covering group $\widehat{S}$ of S acts absolutely irreducible on V. Now using the information on the lower bound for the dimension of the absolutely irreducible representations of finite simple groups, we will get a finite list of cases that we can handle either by constructing the representations or by computer program GAP [10].

As in the almost simple doubly transitive case, we can get a list of maximal subgroups M such that $1_P^G \nleqslant 1_M^G$. In Table 1, we list all the cases when $M \in \mathcal{C}(G)$, Table 2 contains all the cases when $M \in \mathcal{S}(C)$, and in the last table, we list the cases that we have not determined whether or not there is a containment. Notice that we only have a finite number of exceptions in Table 3. Also there is a finite number of cases in Table 2.

For the notation in the tables of Theorem 1.1, the columns 'orbits' give the number of orbits of M on $\mathfrak{E}(V)$ and this is also the number of double cosets of G on P and M. The first columns are the type of the nearly simple group G. The last columns 'Ref' give the references for the result. For example '5.3' means that the case is dealt with in Proposition 5.3.

2. Preliminaries

We adopt the constructions and notation of [21]. Fix a finite field $\mathbb{F}_q$, q a prime power. Let V be an $\mathbb{F}_q$ vector space of dimension n. The *general linear group* $GL(V)$ of V over $\mathbb{F}_q$ is a group of all non-singular $\mathbb{F}_q$-linear transformations of V. The *special linear group* of V over $\mathbb{F}_q$, $SL(V)$, is the group of all elements of $GL(V)$ with determinant 1. The *projective linear groups* $PGL(V)$ and $PSL(V)$ are obtained by factoring out the scalar matrices in the corresponding linear groups. For any subgroup X of $GL(V)$, we write PX or $\overline{X}$ for the corresponding projective group $X/X \cap \mathbb{F}_q^*$.

A map g from V to itself is called an $\mathbb{F}_q$-*semilinear transformation* of V if there is a field automorphism $\sigma(g) \in Aut(\mathbb{F}_q)$ such that for all $v, w \in V$ and $\lambda \in \mathbb{F}_q$,

$$(v + w)g = vg + wg \text{ and } (\lambda v)g = \lambda^{\sigma(g)}(vg) \tag{2.1}$$

The *general semilinear group* of V over $\mathbb{F}_q$, $\mathit{\Gamma L}(V)$ consists of all non-singular $\mathbb{F}_q$-semilinear transformations of V. As $\mathbb{F}_q^* \trianglelefteq \mathit{\Gamma L}(V)$, we can factor out the scalars to get the *projective general semilinear group* $P\mathit{\Gamma L}(V)$. Let κ be a left linear or a quadratic form on V. Observe κ is a map from V^k to $\mathbb{F}_q$ where $k = 1, 2$. Define

$$I(V, \mathbb{F}_q, \kappa) = \{g \in GL(V) \mid \kappa(\mathbf{v}g) = \kappa(\mathbf{v}) \text{ for all } \mathbf{v} \in V^k\};$$
$$S(V, \mathbb{F}_q, \kappa) = I(V, \mathbb{F}_q, \kappa) \cap SL(V);$$
$$\Xi(V, \mathbb{F}_q, \kappa) = \{g \in \mathit{\Gamma L}(V) \mid \kappa(\mathbf{v}g) = \tau(g)\kappa(\mathbf{v})^{\sigma(g)} \text{ for all } \mathbf{v} \in V^k\}$$

where $\tau(g) \in \mathbb{F}_q^*, \sigma(g) \in Aut(\mathbb{F}_q)$, and

$$\Lambda(V, \mathbb{F}_q, \kappa) = \{g \in \Xi(V, \mathbb{F}_q, \kappa) \mid \sigma(g) = 1\}.$$

Define

$$A = \begin{cases} \Xi\langle\iota\rangle & \text{in case } \mathbf{L} \text{ with } n \geqslant 3; \\ \Xi & \text{otherwise;} \end{cases}$$

and

$$\Omega = \begin{cases} \text{certain subgroup of index 2 in S} & \text{in case } \mathbf{O}; \\ S & \text{otherwise;} \end{cases}$$

where ι is an inverse-transpose automorphism of $GL(V, \mathbb{F}_q)$. We get a sequence of groups: $\Omega \leqslant S \leqslant I \leqslant \Lambda \leqslant \Xi \leqslant A$. Note that in [21], $\Xi(V, \mathbb{F}_q, \kappa)$, and $\Lambda(V, \mathbb{F}_q, \kappa)$ are denoted by $\mathit{\Gamma}(V, \mathbb{F}_q, \kappa)$, and $\Delta(V, \mathbb{F}_q, \kappa)$, respectively. For more details, see [21, Chapter 2].

Let $(V, \mathbb{F}_q, Q)$ be a classical orthogonal geometry with q odd, and $dimV = n$. Let $\varepsilon = sgn(Q)$ be the sign of the quadratic form Q. Note that $\varepsilon = \circ$ when n is odd, otherwise, ε is either $+$ or $-$. A square or a non-square in $\mathbb{F}_q^*$ will be denoted by $\square$ or $\boxtimes$, respectively. If W is a non-degenerate subspace of V then we write $sgn(W) = sgn(Q_W)$, where Q_W is the restriction of Q to W. When n is odd, for any non-zero vector x in V, we denote by $S(n, x)$ the number of all vectors $v \in V$ with $Q(v) = Q(x)$. When n is even, for any $\gamma \in \mathbb{F}_q$, we denote by $S^\varepsilon(n, \gamma)$ the number of all vectors $v \in V$ with $Q(v) = \gamma$. For $x \in V \setminus \{0\}$, a one-space with representative x will be called a *point* in V and denoted by $\langle x\rangle$. We now define a type function $\rho = \rho_V$ on $V \setminus \{0\}$ as follows: if x is a singular vector in V, that is, $x \in V \setminus \{0\}$ and $Q(x) = 0$, then $\rho(x) = 0$. If $dimV$ is even, then $\rho(x) = Q(x)$. If $dimV$ is odd, then $\rho(x) = \mathrm{sgn}(x^{\perp})$. Assume that $dimV$ is odd. Let x be a non-singular vector in V. We say x is a *plus vector* if $\rho(x) = +$; and x is a *minus vector* if $\rho(x) = -$. We also say that $\langle x\rangle$ is of plus or minus type according to whether its representative x is a plus or a minus vector. Let $x \in V$ be a non-singular vector with $\rho(x) = \xi$. Define $\mathfrak{E}^{\varepsilon}_{\xi}(V)$ to be the set of all non-singular points of type ξ in V, where $\varepsilon = sgn(Q)$. When $\varepsilon = \circ$, we will write $\mathfrak{E}_{\xi}(V)$ instead of $\mathfrak{E}^{\circ}_{\xi}(V)$.

Lemma 2.1. *Let $(V, \mathbb{F}_q, Q)$ be a classical orthogonal geometry with $dimV$ odd. Two non-singular vectors x, y (two non-singular points $\langle x\rangle, \langle y\rangle$) have the same type if and only if $Q(x) \equiv Q(y)$ $(mod\ (\mathbb{F}_q^*)^2)$. Hence for any non-singular vector z, we have $\mathfrak{E}_{\rho(z)}(V) = \{\langle v\rangle \subseteq V \mid Q(v) \equiv Q(z)\ (mod\ (\mathbb{F}_q^*)^2)\}$. In particular if $q = 3$, then $\mathfrak{E}_{\rho(z)}(V) = \{\langle v\rangle \subseteq V \mid Q(v) = Q(z)\}$.*

Proof. Assume that $Q(x) \equiv Q(y)$ (mod $(\mathbb{F}_q^*)^2$). By [21, Proposition 2.5.4(ii)], $\langle x\rangle$ and $\langle y\rangle$ are isometric. By Witt's lemma, this isometry extends to an isometry g of V such that $\langle x\rangle g = \langle y\rangle$. As $\langle x\rangle, \langle y\rangle$ are non-degenerate, $x^{\perp}g = y^{\perp}$. It follows that $x^{\perp}$ and $y^{\perp}$ are isometric, and hence $sgn(x^{\perp}) = sgn(y^{\perp})$, so that $\rho(x) = \rho(y)$. Now assume that x, y have the same type. By Witt's lemma and [21, Proposition 2.5.4(i)], there exists an isometry between $x^{\perp}$ and $y^{\perp}$. This isometry can extend to an isometry g of V such that $(x^{\perp})g = y^{\perp}$. Since $(x^{\perp})^{\perp} = \langle x\rangle$, and $(y^{\perp})^{\perp} = \langle y\rangle$, $(\langle x\rangle)g = \langle y\rangle$. Thus $xg = \mu y$ for some $\mu \in \mathbb{F}_q^*$. Therefore $Q(x) = Q(xg) = Q(\mu y) = \mu^2 Q(y)$. The other statements are obvious. $\square$

The following lemma will be used to compute the Higman rank 3 parameters for orthogonal groups.

Lemma 2.2. *Let $(V, \mathbb{F}_q, Q)$ be a classical orthogonal geometry with q odd and $\varepsilon = sgn(Q)$.*

(1) *if $dimV = 2k$ and $\gamma \in \mathbb{F}_q$, then* $S^{\varepsilon}(2k, \gamma) = \begin{cases} q^{2k-1} + \varepsilon(q^k - q^{k-1}) & \text{if } \gamma = 0, \\ q^{2k-1} - \varepsilon q^{k-1} & \text{if } \gamma \neq 0; \end{cases}$

(2) *if $dimV = 2k + 1$ and $x \in V - \{0\}$ then $S(2k + 1, x) = q^{2k} + \rho(x)q^k$.*

Proof. Statement (1) follows from [11, Proposition 9.10]. For (2), let $\gamma = Q(x)$ and $\xi = \rho(x)$. Assume first that x is a non-singular vector. Then $V = \langle x\rangle \perp x^{\perp}, \mathrm{sgn}(x^{\perp}) = \xi$ and $\dim(x^{\perp}) = 2k$. For any vector $v \in V$ with $Q(v) = \gamma$, write $v = \varphi x + v_0$, where $\varphi \in \mathbb{F}_q$ and $v_0 \in x^{\perp}$. We have $Q(v_0) = Q(v) - \varphi^2 Q(x) = \gamma(1 - \varphi^2)$. If $\varphi = \pm 1$, then $Q(v_0) = 0$, hence by (1), there are $2S^{\xi}(2k, 0) = 2(q^{2k-1} + \xi(q^k - q^{k-1}))$ such v. If $\varphi \neq \pm 1$, then $Q(v_0) = \gamma(1 - \varphi^2) \neq 0$, hence by (1), again, there are $(q - 2)S^{\xi}(2k, \gamma(1 - \varphi^2)) = (q - 2)(q^{2k-1} - \xi q^{k-1})$ such v. Thus $S(2k + 1, x) = 2S^{\xi}(2k, 0) + (q - 2)S^{\xi}(2k, \gamma(1 - \varphi^2)) = q^{2k} + \xi q^k$. Assume that x is a singular vector. Observe that V always contains a non-singular vector y. Let $\eta = \rho(y)$ and $\mu = Q(y)$. We have $V = \langle y\rangle \perp y^{\perp}$, $sgn(y^{\perp}) = \eta, Q(y) = \mu \in \mathbb{F}_q^*$ and $\dim(y^{\perp}) = 2k$. For any $v \in V$ with $Q(v) = 0$, write $v = \varphi y + v_0$, where $\varphi \in \mathbb{F}_q$ and $v_0 \in y^{\perp}$. Then $Q(v_0) = Q(v) - \varphi^2 Q(y) = -\mu\varphi^2$. If $\varphi = 0$, then $Q(v_0) = 0$, hence by (1), there are $S^{\eta}(2k, 0) = q^{2k-1} + \eta(q^k - q^{k-1})$ such v. If $\varphi \neq 0$, then $Q(v_0) = -\mu\varphi^2 \neq 0$, hence by (1), again, there are $(q-1)S^{\eta}(2k, -\mu\varphi^2) = (q-1)(q^{2k-1} - \eta q^{k-1})$ such v. Thus $S(2k + 1, x) = S^{\eta}(2k, 0) + (q - 1)S^{\eta}(2k, -\mu\varphi^2) = q^{2k}$. The proof is complete. $\square$

By the classification of the finite simple groups, every non-abelian finite simple group is either an alternating group $A_n, n \geqslant 5$, a finite simple group of Lie type, or

TABLE 4. Upper bounds for the size of $Aut(L)$.

L	$f(L)$	L	$f(L)$
$L_n(q), n \geqslant 3$	q^{n^2}	$E_8(q)$	q^{249}
$PSp_{2n}(q)$	q^{2n^2+n+1}	$F_4(q)$	q^{53}
$U_n(q), n \geqslant 3$	q^{n^2}	${}^2E_6(q)$	q^{79}
$P\Omega^+_{2n}(q), n \neq 4$	q^{2n^2-n+1}	$G_2(q)$	q^{15}
$P\Omega^+_8(q)$	$3q^{29}$	${}^3D_4(q)$	$3q^{29}$
$P\Omega^-_{2n}(q)$	q^{2n^2-2n+3}	${}^2F_4(q)$	q^{27}
$\Omega_{2n+1}(q)$	q^{2n^2+n+1}	$Sz(q)$	q^6
$E_6(q)$	q^{79}	${}^2G_2(q)$	q^8
$E_7(q)$	q^{134}	$P\Omega^-_{2n}(q)$	$2q^{2n^2-2n+2}$

one of the 26 sporadic simple groups. The next lemma gives an upper bound for the full automorphism groups of the non-abelian finite simple groups of Lie type. The proof will be omitted.

Lemma 2.3. *If L is a finite simple group of Lie type, then $|Aut(L)| \leqslant f(L)$, where $f(L)$ is given in Table 4.*

Let G be a finite group and P, M be subgroups of G. Denote by $M\backslash G/P$, a set of representatives for the double cosets of G on P and M. Let $\mathfrak{C} = G/P$, the right cosets of G on P. Then M acts on $\mathfrak{C}$ by right multiplication.

Lemma 2.4. *Let M, P be subgroups of a finite group G. Then*
(i) M has $|M\backslash G/P|$ orbits on G/P, where M acts on G/P by right multiplication.
(ii) $(1_M^G, 1_P^G) = |M\backslash G/P|$.

Proof. (*i*) Suppose that $M\backslash G/P = \{x_1, \dots, x_k\}$. If $i, j \in \{1, \dots, k\}$ with $i \neq j$, then $Px_iM \cap Px_jM = \emptyset$. Clearly Px_iM are distinct orbits of M on G/P. As $G = \cup_{i=1}^k Px_iM$, $\{Px_iM\}_{i=1}^k$ is a complete set of orbits of M on G/P.

(*ii*) By [17, Corollary 5.5], the number of orbits of M on G/P is the inner product $(1_M, (1_P^G)_M) = \frac{1}{|M|}\sum_{m\in M} 1_P^G(m)$, and so $(1_M, (1_P^G)_M) = |M\backslash G/P|$ by (*i*). Now by the Frobenius reciprocity [17, Lemma 5.2], we have $(1_M^G, 1_P^G) = (1_M, (1_P^G)_M)$, so that $(1_M^G, 1_P^G) = |M\backslash G/P|$ as required. □

Note that (*ii*) could follow easily from the Mackey's formula.

For the representation theory of symmetric groups and the Mullineux conjecture, we refer to [19], [7] or [8]. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$ be a partition of n, and let p be a prime. We denote by $[\lambda]$ the Young diagram of the partition λ, which consists of n nodes placed in decreasing rows. The *rim* of $[\lambda]$ is its south-east border. The *p-rim* of $[\lambda]$ is defined as follows: Beginning at the top right-hand corner of $[\lambda]$, the first p nodes of the rim are in the p-rim. Then skip to the next row, and take the next p nodes of the rim. Continue until we reach the end of the rim. The last p-segment may contain fewer than p nodes (see [7] or [8]). Let h_1 be the number of nodes in the p-rim of λ, and let r_1 be the number of rows in λ. Delete the p-rim and repeat the process to get $h_1, r_1, \dots, h_k, r_k$, where $h_{k+1} = r_{k+1} = 0$, but $h_k \neq 0 \neq r_k$. The *Mullineux symbol* is a $2 \times k$ matrix,

$$M(\lambda) = \begin{pmatrix} h_1 & h_2 & \dots & h_k \\ r_1 & r_2 & \dots & r_k \end{pmatrix}.$$

Now the p-regular partition $m(\lambda)$ of n is defined via

$$M(m(\lambda)) = \begin{pmatrix} h_1 & h_2 & \dots & h_k \\ s_1 & s_2 & \dots & s_k \end{pmatrix},$$

where

$$\varepsilon_i = \begin{cases} 0, & \text{if } p \mid h_i \\ 1, & \text{if } p \nmid h_i \end{cases}$$

and $s_i = h_i - r_i + \varepsilon_i$. Note that the partition $m(\lambda)$ can be reconstructed from the Mullineux symbol $M(m(\lambda))$. The following notation will be useful if we just want to know whether a given partition is fixed under the Mulllineux map. The *p-modular Frobenius symbol* for λ, denoted by $Fr_p(\lambda)$, is a $3 \times k$ matrix

$$Fr_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_k \end{pmatrix}$$

where $a_i = h_i - r_i$ and $b_i = r_i - \varepsilon_i$. If λ has p-modular Frobenius symbol $Fr_p(\lambda)$ as constructed above then the Mullineux map m is defined by

$$Fr_p(m(\lambda)) = \begin{pmatrix} b_1 & b_2 & \dots & b_k \\ a_1 & a_2 & \dots & a_k \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_k \end{pmatrix}$$

which means that we interchange the first two rows of $Fr_p(\lambda)$. Therefore we see that λ is a fixed point of the Mullineux map if and only if the first two rows of $Fr_p(\lambda)$ are the same (see [7]). Denote by sgn_n, the sign character of S_n, which takes value 1 at even permutation and -1 at odd permutation. Now if λ is a p-regular partition of n then $D^\lambda \bigotimes sgn_n = D^{m(\lambda)}$ (see [8]), where D^λ denote the irreducible S_n-module in characteristic p, corresponding to the p-regular partition λ. Let λ be a 3-regular partition of n with $\lambda_1 \geqslant n-2$. We will show that λ is rarely a fixed point of the Mullineux map.

Lemma 2.5. *Assume that $p = 3, n \geqslant 5$ and λ is a p-regular partition of n. Suppose that $\lambda_1 \geqslant n-2$. Then $m(\lambda) = \lambda$ if and only if $5 \leqslant n \leqslant 6$ and $\lambda = (n-2, 1^2)$.*

Proof. As $\lambda_1 \geqslant n-2$, $\lambda_1 = n, n-1$ or $n-2$. It follows that $\lambda = (n), (n-1, 1), (n-2, 2)$ or $(n-2, 1^2)$. If one of the first three cases holds then the result follows from [22, Lemma 1.8]. Assume that $\lambda = (n-2, 1^2)$. We first compute the Mullineux symbol $M(\lambda)$ of λ. We have

$$M(\lambda) = \begin{pmatrix} 5 & 3 & \dots & 3 & a \\ 3 & 1 & \dots & 1 & 1 \end{pmatrix},$$

where 3, and so 1, occurs t times with $t = \left[\frac{n-2}{3}\right] - 1$, and $0 \leqslant a = n-2-3(t+1) \leqslant 2$. Hence

$$Fr_3(\lambda) = \begin{pmatrix} 2 & 2 & \dots & 2 & a-1 \\ 2 & 1 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

If $t \geqslant 1$, or equivalently $n \geqslant 8$, then clearly, the first two rows of $Fr_3(\lambda)$ cannot be equal, so that $\lambda \neq m(\lambda)$. Thus $5 \leqslant n \leqslant 7$ and $t = 0$. Then

$$Fr_3(\lambda) = \begin{pmatrix} 2 & a-1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Observe that the first two rows of $Fr_3(\lambda)$ are equal if and only if $a = 0$ or $a = 1$. Since $t = 0$, $a = n - 2 - 3 = n - 5$, so that $n = 5$ or 6. □

We next prove a gap result between the minimal module and the second minimal module for alternating groups in characteristic 3. For $k \geqslant 1$, denote by $R_n(k)$ the set of irreducible S_n-modules D such that $D \cong D^\lambda$ or $D \cong D^{m(\lambda)}$ for some p-regular partition $\lambda \vdash n$, with $\lambda_1 \geqslant n - k$.

Lemma 2.6. *Let $\mathbb{F}$ be a splitting field for A_n of characteristic $p = 3$. Suppose that $n \geqslant 12$ and V is an irreducible $\mathbb{F}A_n$-module with $dimV > n$. Then $dimV \geqslant (n^2 - 5n + 2)/2$.*

Proof. It follows from [7, Theorem 2.1] that $V = D^\lambda \downarrow_{A_n}$ with $m(\lambda) \neq \lambda$ or $V = D^\lambda_\pm$, where $m(\lambda) = \lambda$. Let $U = D^\lambda$ for the partition λ obtained above. By [29, Proposition 2.2], either $\dim U > (n-2)(n-3)$, or $U \in R_n(2)$. Observe that $dimV = dimU$ if $m(\lambda) \neq \lambda$ and if $m(\lambda) = \lambda$ then $dimV = \frac{1}{2}dimU$. Thus if $dimU > (n-2)(n-3)$, then clearly, $\dim V \geqslant \frac{1}{2}dimU > \frac{1}{2}(n-2)(n-3) > \frac{1}{2}(n^2 - 5n + 2)$. Therefore we can assume $U \in R_n(2)$. Hence there exists a 3-regular partition μ with $\mu(1) \geqslant n-2$ such that $\lambda = \mu$ or $\lambda = m(\mu)$. As $n > 10$ and $dimV > n$, it follows from [19, Theorem 6] that μ is neither (n) nor $(n-1, 1)$, and so $\mu = (n-2, 2)$, or $(n-2, 1^2)$. By Lemma 2.5, μ is not fixed under the Mullineux map. Also, as the Mullineux map is an involutory map, $m(m(\mu)) = \mu \neq m(\mu)$. We conclude that D^μ and $D^{m(\mu)}$ are irreducible upon restriction to A_n. Since $dimD^{m(\mu)} = dimD^\mu \otimes sgn_n = dimD^\mu$, we have $dimV = dimD^\lambda = dimD^\mu$. Finally, the result follows from [18, Theorem 24.1, 24.15]. □

3. Higman rank 3 parameters and the equation

Let G be a finite group acting transitively on a non-empty finite set $\mathfrak{E}$. Let P be a stabilizer of a point $x \in \mathfrak{E}$ in G. Recall that the action is primitive if and only if P is maximal in G. Also the rank of G is the number of orbits of P on $\mathfrak{E}$. Now suppose that G is of even order and acts primitively rank 3 on $\mathfrak{E}$. So P has exactly three orbits on $\mathfrak{E}$, namely, $\{x\}$, $\Delta(x)$ and $\Gamma(x)$. Define $k = |\Delta(x)|$, $l = |\Gamma(x)|$ and

$$|\Delta(x) \cap \Delta(y)| = \begin{cases} \mu & \text{if } y \in \Gamma(x) \\ \lambda & \text{if } y \in \Delta(x) \end{cases}$$

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} \lambda_1 & \text{if } y \in \Gamma(x) \\ \mu_1 & \text{if } y \in \Delta(x). \end{cases}$$

Suppose that $k \leqslant l$.

Lemma 3.1. ([13, Lemma 5, 7]). *Let G act primitively rank 3 on $\mathfrak{E}$. Then*
(i) $|\mathfrak{E}| = k + l + 1$;
(ii) $\mu l = k(k - 1 - \lambda)$;
(iii) $D = (\lambda - \mu)^2 + 4(k - \mu)$ *is a square;*
(iv) $\lambda_1 = l - k + \mu - 1$;

(v) $\mu_1 = l - k + \lambda + 1$.

Let V be the permutation module for G on $\mathfrak{E}$ over $\mathbb{C}$, hence $\mathfrak{E}$ is a basis for V. Further Δ and Γ can be viewed as linear maps on V, via the corresponding $x \mapsto \sum_{y\in\Delta(x)} y$ and $x \mapsto \sum_{y\in\Gamma(x)} y$, for $x \in \mathfrak{E}$, and extending linearly. We have that $\Sigma_{y\in\mathfrak{E}} y$ is an eigenvector for Δ and Γ, with eigenvalues k and l, respectively. Other eigenvalues for Δ and Γ are as follows:

Lemma 3.2. ([13, Lemma 6]). *The eigenvalues different from k of Δ are:*

$$s = \frac{\lambda - \mu + \sqrt{D}}{2} \quad \textit{and} \quad t = \frac{\lambda - \mu - \sqrt{D}}{2}.$$

Lemma 3.3. ([1, 1.4.3]) *For $r \in \{s, t\}$, the eigenvalues for Γ are $-(r + 1)$.*

Let V_r, $r = s$ or $r = t$, be the irreducible modules for G on V which is the eigenspace for Δ with eigenvalue r. Let $f_r = dim(V_r)$.

Lemma 3.4. ([13]). *We have*

$$f_s = \frac{k + t(k + l)}{t - s} \textit{ and } f_t = \frac{k + s(k + l)}{s - t}.$$

Let M be any subgroup of G. Fix $x \in \mathfrak{E}$, and let $P = G_x$. By identifying $\mathfrak{E}$ with G/P, x corresponds to the coset P in G/P. Consider the action of G on the right cosets G/M and form the permutation module V_M. Denote by y the coset M as a point in G/M. Define

$$d = d_x = |xM \cap \Delta(x)| \text{ and } c = c_x = |xM \cap \Gamma(x)|.$$

As G acts transitively on $\mathfrak{E}$ and G/M, $xG = \mathfrak{E}$ and $G/M = yG$. Define $\alpha : V \to V_M$ by $\alpha(xg) = \Sigma_{p\in P} ypg$ for $g \in G$ and $\beta : V_M \to V$ by $\beta(yg) = \Sigma_{m\in M} xmg$ for $g \in G$. Then α and β are $\mathbb{C}G$ maps and $\theta = \frac{1}{|P||P\cap M|} \beta \circ \alpha \in End_{\mathbb{C}G}(V)$. The map θ can be written in terms of the linear maps Δ and Γ as follows:

Lemma 3.5. ([1, 2.1]). *We have*

$$\theta = I + \frac{d\Delta}{k} + \frac{c\Gamma}{l}.$$

Proof. For any $g \in G$, we have

$$\begin{aligned} \theta(xg) &= \frac{1}{|P||P \cap M|} \beta(\alpha(xg)) \\ &= \frac{1}{|P||P \cap M|} \textstyle\sum_{p\in P} \beta(ypg) \\ &= \frac{1}{|P||P \cap M|} \textstyle\sum_{p\in P, m\in M} xmpg. \end{aligned}$$

Let $\mathcal{O}$ be one of the three orbits of P on $\mathfrak{E}$ and $v_{\mathcal{O}} = \sum_{u\in\mathcal{O}} u \in V$. As P acts on $\mathcal{O}$, $xmp \in \mathcal{O}$ if and only if $xm \in \mathcal{O}$, in which case since P is transitive on $\mathcal{O}$, $\sum_{p\in P} xmp = \frac{|P|}{|\mathcal{O}|} v_{\mathcal{O}}$. Moreover there are $|M_x| = |P \cap M|$ elements in M fixing x and $|P \cap M|d$, $|P \cap M|c$ elements $m \in M$ with $xm \in \mathcal{O}$ for $\mathcal{O} = \Delta, \Gamma$, respectively. Therefore

$$\theta(xg) = (v_x + \frac{dv_\Delta}{k} + \frac{cv_\Gamma}{l})(x)g = (I + \frac{d\Delta}{k} + \frac{c\Gamma}{l})(xg).$$

This proves the lemma. ∎

For $r = s, t$, let π_r be the projection of V on V_r.

Lemma 3.6. ([1, 2.2, 2.3]). *Let $r = s$ or t, then*
(1) $\theta \circ \pi_r = 0$ if and only if $V_r \leqslant ker(\theta)$;
(2) If $\theta \circ \pi_r \neq 0$ then $\alpha : V_r \to V_M$ is an injective $\mathbb{C}G$ map;
(3) For $r = s, t : \theta \circ \pi_r = 0$ if and only if

$$1 + \frac{dr}{k} = \frac{(r+1)c}{l}. \tag{3.1}$$

Proof. Statement (1) is clear as $\pi_r(V) = V_r$. By Lemma 3.4, V_s is not $\mathbb{C}G$-isomorphic to V_t, so that $\theta : V_r \to V_r$. As V_r is an irreducible $\mathbb{C}G$-module, θ is an isomorphism if θ is non-zero. Thus (2) follows from (1). For (3), let $r = s, t$ and $v \in V$. Let $\Sigma = \Delta$ or Γ. Then $(\Sigma \circ \pi_r)(v) = e(\Sigma, r)\pi_r(v)$, where $e(\Sigma, r)$ is the eigenvalue of Σ on V_r. Thus $\Sigma \circ \pi_r = e(\Sigma, r)\pi_r$. From definition, $e(r, \Delta) = r$ and by Lemma 3.3, $e(r, \Gamma) = -(r + 1)$. Therefore by Lemma 3.5, we obtain:

$$\theta \circ \pi_r = (I + \frac{d\Delta}{k} + \frac{c\Gamma}{l}) \circ \pi_r = (1 + \frac{dr}{k} - \frac{(r+1)c}{l})\pi_r.$$

Thus $\theta \circ \pi_r = 0$ if and only if $1 + \frac{dr}{k} - \frac{(r+1)c}{l} = 0$. This finishes the proof. □

As G is a primitive rank 3 group on $\mathfrak{E}$ with P the stabilizer of a point x in $\mathfrak{E}$, the permutation character 1_P^G has a decomposition $1_P^G = 1 + \chi_s + \chi_t$, where 1 is the trivial character, and χ_s, χ_t are irreducible characters of G, afforded by the irreducible modules V_s, V_t, with degrees f_s, f_t, respectively. From definition, $1_P^G \leqslant 1_M^G$ if $1_M^G - 1_P^G$ is a character of G. This is equivalent to $(\chi_r, 1_M^G) > 0$ for any $r \in \{s, t\}$. By Lemma 3.6, for $r \in \{s, t\}$, $(\chi_r, 1_M^G) = 0$ if and only if equation (3.1) holds for any M-orbits in $\mathfrak{E}$. Note that the parameters c and d depend on x, or equivalently, on the conjugate of P. When we pick a different conjugate of P, parameters c and d change. Thus for $r \in \{s, t\}$, if equation (3.1) does not hold for some point $x \in \mathfrak{E}$ or some conjugate of P, then by Lemma 3.6, there is an injective $\mathbb{C}G$ map from V_r to V_M and hence $1_P^G \leqslant 1_M^G$. Otherwise we need to change to different conjugate of P or different point in $\mathfrak{E}$. See Proposition 5.13 for such an example.

The following result will be used frequently to show that there is no containment if M has at most 2 orbits on $\mathfrak{E}$.

Corollary 3.7. *Let G be a primitive rank 3 group acting on a finite set $\mathfrak{E}$. Let P be the stabilizer of a point in $\mathfrak{E}$, and let M be any subgroup of G. If M has at most two orbits on $\mathfrak{E}$, then $1_P^G \nleqslant 1_M^G$.*

Proof. By way of contradiction, suppose that $1_P^G \leqslant 1_M^G$. Then $\phi = 1_M^G - 1_P^G$ is a character of G. Since ϕ and 1_P^G are characters of G, we have that

$$(1_P^G, 1_M^G) = (1_P^G, \phi + 1_P^G) = (1_P^G, 1_P^G) + (1_P^G, \phi) = 3 + (1_P^G, \phi) \geqslant 3,$$

By Lemma 2.4, $(1_P^G, 1_M^G)$ is the number of orbits of M on G/P. Now, by identifying $\mathfrak{E}$ with G/P, M has $(1_P^G, 1_M^G) \geqslant 3$ orbits on $\mathfrak{E}$, a contradiction. □

4. Main Hypothesis and Notations

From now on, we assume the following set up. Let $L = \Omega_{2m+1}(3)$ with $m \geqslant 2$. Let V be the natural module for L over $\mathbb{F} = \mathbb{F}_3$. Denote by $\mathfrak{E}_\xi(V)$ an L-orbit of non-singular points of type ξ in V. Let G be a nearly simple primitive rank 3 group of type L acting on $\mathfrak{E}_\xi(V)$. Observe that $L \trianglelefteq G \leqslant I(V)$, where $I(V)$ is the full isometric group of V. Denote by P the stabilizer of a point in $\mathfrak{E}_\xi(V)$. The letter M will be reserved for the maximal subgroup of G. If $M \in \mathcal{C}(G)$ and X is a group satisfying $\Omega(V) \leqslant X \leqslant \Xi(V)$, then there exists a subgroup $H \in \mathcal{C}(\Xi)$ such that $H \cap G = M$ and the subgroup $H \cap X \in \mathcal{C}(X)$ is called the X-*associate* of M and is denoted by M_X. If $M \in \mathcal{S}(G)$ then we denote the socle of M by S. Then S is a non-abelian finite simple group and the full covering group $\widehat{S}$ of S acts absolutely irreducible on V and is not realizable over a proper subfield of $\mathbb{F}$. Moreover $\widehat{S}$ fixes a non-degenerate quadratic form on V so that the Frobenius-Schur indicator $ind(V) = +$.

5. Character containment for nearly simple groups of type L

5.1. **Higman rank 3 parameters for L.** We now assume the hypotheses and notation in Section 4 with $L = \Omega_{2m+1}(3), m \geqslant 2$. There are two types of non-singular points in V, namely $+$ and $-$ points. For $\xi \in \{\pm\}$, denote by $\mathfrak{E}_\xi(V)$ the set of all non-singular points of type ξ. For any $\langle x\rangle \in \mathfrak{E}_\xi(V)$, define

$$\Delta(x) = \mathfrak{E}_\xi(V) \cap x^\perp \text{ and } \Gamma(x) = \mathfrak{E}_\xi(V) \cap (V - x^\perp - \{\langle x\rangle\}).$$

Then P has exactly three orbits $\{\langle x\rangle\}, \Delta(x)$ and $\Gamma(x)$ on the set $\mathfrak{E}_\xi(V)$. Recall the Higman rank 3 parameters defined in Section 3. For $\xi = \pm$, we write $\xi 1$ to denote $+1$ or -1 when $\xi = +$ or $\xi = -$, respectively.

Lemma 5.1. *Let $\xi \in \{\pm\}$ and $\langle x\rangle \in \mathfrak{E}_\xi(V)$. Then*
(i) $|\mathfrak{E}_\xi(V)| = \frac{1}{2}3^m(3^m + \xi 1)$;
(ii) $k = \frac{1}{2}3^{m-1}(3^m - \xi 1)$;
(iii) $l = (3^m - \xi)(3^{m-1} + \xi 1)$;
(iv) $\mu = \lambda = \frac{1}{2}3^{m-1}(3^{m-1} - \xi 1)$;
(v) $\sqrt{D} = 2 \cdot 3^{m-1}$;
(vi) $s = 3^{m-1}$;
(vii) $t = -3^{m-1}$.

Proof. Let $\gamma = Q(x)$. We have $\rho(x) = sgn(x_V^\perp) = \xi$. By Lemma 2.1, we obtain $\mathfrak{E}_\xi(V) = \{\langle v\rangle \subseteq V \mid Q(v) = \gamma\}$. Observe that each point $\langle v\rangle$ has 2 representatives v and $-v$, by Lemma 2.2, $|\mathfrak{E}_\xi(V)| = \frac{1}{2}S(2m+1, x) = \frac{1}{2}(3^{2m} + \xi 3^m) = \frac{1}{2}3^m(3^m + \xi 1)$, which gives (i).

From definition we get

$$k = |\Delta(x)| = |\mathfrak{E}_\xi(V) \cap x^\perp| = |\{\langle v\rangle \subseteq x^\perp \mid Q(v) = Q(x)\}|.$$

By Lemma 2.2 again, $k = \frac{1}{2}S^\xi(2m, \gamma) = \frac{1}{2}S^\xi(2m, \gamma) = \frac{1}{2}(q^{2m-1} - \xi q^{m-1})$. The parameter l can be computed from the relation $1 + k + l = |\mathfrak{E}_\xi(V)|$. This proves (ii) and (iii).

To compute λ, take $\langle y\rangle \in \Delta(x)$. Then $\rho(y) = \rho(x) = \xi, (x, y) = 0$ and $Q(y) = Q(x)$. We have

$$\lambda = |\Delta(x) \cap \Delta(y)| = \frac{1}{2}|\{v \in x^\perp \cap y^\perp \mid Q(v) = Q(x)\}| = \frac{1}{2}S(2m-1, z),$$

where $z \in W := x^\perp \cap y^\perp = \langle x, y\rangle^\perp$ with $Q(z) = Q(x)$. We need to determine $\rho_W(z) = \mathrm{sgn}(z_W^\perp)$. Since $x^\perp = \langle y\rangle \perp (y^\perp \cap x^\perp) = \langle y\rangle \perp W$ and $W = z_W^\perp \perp \langle z\rangle$, we deduce that $x^\perp = \langle y, z\rangle \perp z_W^\perp$, where $\dim(z_W^\perp) = 2m-2, \dim W = 2m-1$ and $\dim\langle y, z\rangle = 2$. As $D\langle y, z\rangle = det(\mathrm{diag}(-\gamma, -\gamma)) = \square$, by [21, Proposition 2.5.13], $\mathrm{sgn}\langle y, z\rangle = (-)^1 = -$. It follows from [21, Proposition 2.5.10] that $\mathrm{sgn}(z_W^\perp) =$

$-\mathrm{sgn}(x^{\perp}) = -\xi$. Thus by Lemma 2.2, $\lambda = \frac{1}{2}(3^{2m-2} - \xi 3^{m-1})$, which gives (iv). The remaining parameters follow from Lemmas 3.1 and 3.2. □

Corollary 5.2. *Let M be a subgroup of G and $\xi \in \{\pm\}$. Suppose that equation (3.1) holds for some $r \in \{s,t\}$, and for some M-orbit $\langle x\rangle M$ with $x \in \mathfrak{E}_{\xi}(V)$. Then*
(i) If $(\xi, r) = (+, s)$ or $(-, t)$ then equation (3.1) has the form

$$c - 2d = \xi 3^m - 1. \tag{5.1}$$

(ii) If $(\xi, r) = (+, t)$ or $(-, s)$ then equation (3.1) has the form

$$(\xi 3^{m-1} + 1)(\xi 3^m - 1 + c - 2d) = 2c \tag{5.2}$$

(iii) If $m \geqslant 2$ then

$$1 + c + d \geqslant \frac{3^m + 1}{2} \geqslant 3^{m-1}. \tag{5.3}$$

Proof. From definitions we have $c \geqslant 0$ and $d \geqslant 0$. Let $A = 1 + c + d$. Multiply both sides of equation (3.1) by l, we have $(r+1)c = l + drl/k$. Subtracting drl/k from both sides,

$$(r+1)c - \frac{drl}{k} = l. \tag{5.4}$$

(1) If $\xi = +$, $r = s$, then by Lemma 5.1, we have $r = 3^{m-1}, l = (3^m - 1)(3^{m-1} + 1)$, and $l/k = 2(3^{m-1} + 1)/3^{m-1}$. From (5.4),

$$(3^{m-1} + 1)c - 2d(3^{m-1} + 1) = (3^m - 1)(3^{m-1} + 1).$$

Dividing both sides by $3^{m-1} + 1$, we get $c - 2d = 3^m - 1 = \xi 3^m - 1$. Hence $c = 2d + 3^m - 1$. Thus $A = 3^m + 3d \geqslant 3^m \geqslant (3^m + 1)/2$.

(2) If $\xi = -$, $r = t$, then by Lemma 5.1, $r = -3^{m-1}, l = (3^m + 1)(3^{m-1} - 1)$, and $l/k = 2(3^{m-1} - 1)/3^{m-1}$. From (5.4),

$$(-3^{m-1} + 1)c + 2d(3^{m-1} - 1) = (3^m + 1)(3^{m-1} - 1).$$

Dividing both sides by $3^{m-1} - 1$, we get $c - 2d = -3^m - 1 = \xi 3^m - 1$. In this case, $2d = 3^m + 1 + c$. Since $c \geqslant 0$, $2A = 2 + 2c + 3^m + 1 + c \geqslant 3^m + 3 > 3^m + 1$.

(3) If $\xi = +$, $r = t$, then by Lemma 5.1, $r = -3^{m-1}, l = (3^m - 1)(3^{m-1} + 1)$, and $l/k = 2(3^{m-1} + 1)/3^{m-1}$. From (5.4),

$$(-3^{m-1} + 1)c + 2d(3^{m-1} + 1) = (3^m - 1)(3^{m-1} + 1)$$

or

$$2c = (3^{m-1} + 1)(3^m - 1 + c - 2d) = (\xi 3^{m-1} + 1)(\xi 3^m - 1 + c - 2d),$$

and

$$2d(3^{m-1} + 1) = (3^{m-1} + 1)(3^m - 1) + (3^{m-1} - 1)c.$$

As $c \geqslant 0$,

$$2d(3^{m-1} + 1) \geqslant (3^{m-1} + 1)(3^m - 1),$$

or $2d \geqslant 3^m - 1$. Now $2A = 2 + 2c + 2d \geqslant 2 + 2c + 3^m - 1 \geqslant 3^m + 1$.

(4) If $\xi = -$, $r = s$, then by Lemma 5.1, $r = 3^{m-1}, l = (3^m + 1)(3^{m-1} - 1)$, and $l/k = 2(3^{m-1} - 1)/3^{m-1}$. From (5.4),

$$2c = (3^{m-1} - 1)(3^m + 1 - c + 2d) = (\xi 3^{m-1} + 1)(\xi 3^m - 1 + c - 2d).$$

Therefore

$$(3^{m-1} + 1)c = (3^{m-1} - 1)(3^m + 1) + 2d(3^{m-1} - 1).$$

As $d \geqslant 0$, we have $c \geqslant (3^{m-1}-1)(3^m+1)/(3^{m-1}+1)$. Thus

$$A \geqslant 1 + (3^{m-1}-1)(3^m+1)/(3^{m-1}+1) = (3^{2m-1}-3^{m-1})/(3^{m-1}+1).$$

Since

$$\frac{3^{2m-1}-3^{m-1}}{3^{m-1}+1} - \frac{3^m+1}{2} = \frac{(3^m-1+3^{m+1}(3^{m-2}-1)}{2(3^{m-1}+1)} \geqslant 0,$$

we obtain $A \geqslant (3^m+1)/2 > 3^{m-1}$. □

5.2. **Permutation characters of maximal subgroups in $\mathcal{C}$.** By [21, Proposition 2.6.2], $A = I = S \times \langle -1 \rangle$ and $S = \Omega\langle r_\square r_\boxtimes \rangle$. Let $M \in \mathcal{C}(G)$ be a maximal subgroup of G. Let $M_\Xi \in \mathcal{C}(\Xi)$ be such that $M = G \cap M_\Xi$. Then $M_I = M_\Delta = M_\Xi$, and $M_\Omega \leqslant M \leqslant M_I$.

5.2.1. *The reducible subgroups $\mathcal{C}_1$.* The reducible subgroups in $\mathcal{C}_1(\Xi)$ are all the groups M_Ξ of the forms $N_\Xi(W)$, where $dimW = \alpha$, $1 \leqslant \alpha \leqslant 2m+1$ and W is either non-degenerate or totally singular. The corresponding subgroups are of type $O_\alpha^{\varepsilon_1}(3) \perp O_{2m+1-\alpha}^{\varepsilon_2}(3)$ or P_α, where $\varepsilon_1 = sgn(W), \varepsilon_2 = sgn(W^\perp)$. The subgroup M_Ξ is maximal in Ξ except when M_Ξ is of type $O_2^+(3) \perp O_{2m-1}(3)$, as it is contained in the subgroups of type $O_1(3) \perp O_{2m}^\pm(3)$.

Proposition 5.3. *Assume M is of type $O_\alpha^{\varepsilon_1}(3) \perp O_{2m+1-\alpha}^{\varepsilon_2}(3)$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless M is of type $O_1(3) \perp O_{2m}^{\varepsilon_2}(3)$. In this case M is in Table 1.*

Proof. As M is of type $O_\alpha^{\varepsilon_1}(3) \perp O_{2m+1-\alpha}^{\varepsilon_2}(3)$, there exists a non-degenerate subspace $W \leqslant V$ of dimension α such that $M = N_G(W)$. Put $W_1 = W, W_2 = W^\perp, \varepsilon_i = sgn(W_i)$. Write $X_i = X(W_i)$, where X ranges over the symbols Ω, S and I. By [21, Lemma 4.1.1], we have $M_I = I_1 \times I_2$ and $\Omega_1 \times \Omega_2 \leqslant M_\Omega$. Without loss of generality, we can assume $\dim W_1 = 2b+1$, where $0 \leqslant b < m$. If $b = 0$, then the proposition holds. Assume $b \geqslant 1$. Then W_1 contains non-singular points of both types. Let $\langle x_i \rangle, i = 1, 2$ be non-singular points in W_1 of different types. By [21, Lemma 2.10.5], $\langle x_i \rangle \Omega_1 = \langle x_i \rangle I_1, i = 1, 2$. Moreover, as I_2 centralizes $x_i, i = 1, 2$, we have $\langle x_i \rangle M_\Omega = \langle x_i \rangle M_I$, so that $\langle x_i \rangle M_\Omega = \langle x_i \rangle M$. Thus it is sufficient to compute parameters c, d for subgroup M_Ω in L. Let $x \in \{x_1, x_2\}$, $\eta = \rho_{W_1}(x)$ and $\xi = \rho_V(x)$. Since $\langle x \rangle M_\Omega = \mathfrak{E}_\eta(W_1)$, it follows that $c = l(W_1)$ and $d = k(W_1)$, the parameters for $\mathfrak{E}_\eta(W_1)$. By Lemma 5.1, $d = \frac{1}{2}3^{b-1}(3^b - \eta 1)$, $c = (3^b - \eta 1)(3^{b-1} + \eta 1)$, and so $c - 2d = \eta 3^b - 1$. Suppose that equation (3.1) holds for some $r = s, t$ and any M-orbits on $\mathfrak{E}_\xi(V)$. If (5.1) holds then $\eta 3^b - 1 = \xi 3^m - 1$. This implies that $b = m$, a contradiction. Thus (5.2) holds. Then $(3^{m-1} + \xi 1)(3^m - \xi 2 + \xi\eta 3^b) = 2(3^b - \eta 1)(3^{b-1} + \eta 1)$. Observe that $m - 1 \geqslant b$. Assume first that $m - 1 = b$ and $\xi = -\eta$. We have $3^{m-1} + \xi 1 = 3^b - \eta 1$ and $3^m - \xi 2 + \xi\eta 3^b = 3^{b+1} + \eta 2 - 3^b = 2(3^b + \eta 1)$. Clearly $2(3^b + \eta 1) > 2(3^{b-1} + \eta 1)$, and hence equation (5.2) does not hold in this case. Next assume that $m - 1 = b$ and $\xi = \eta$. Then $(3^{m-1} + \xi 1)(3^m - \xi 2 + \xi\eta 3^b) = (3^b + \xi 1)(3^{b+1} - \xi 2 + 3^b) = 2(3^b + \eta 1)(2 \cdot 3^b - \eta 1) > 2(3^{b-1} + \eta 1)(3^b - \eta 1)$, a contradiction. Finally assume that $m - 1 > b$ so that $m - 1 \geqslant b + 1$. Then $3^{m-1} + \xi \geqslant 3^{b+1} + \xi 1 > 3^b - \eta 1$ and $3^m - \xi 2 + \xi\eta 3^b \geqslant 9 \cdot 3^b - \xi 2 - 3^b = 8 \cdot 3^b - \xi 2 > 2(3^{b-1} + \eta 1)$. Multiplying these two inequalities side by side will lead to a contradiction. Thus equation (5.2) does not hold. Therefore $b = 0$ and so W is a point or a hyperplane. □

Let $\mathbb{F}_q$ be a finite field of size q and let W be a totally singular subspace of V of dimension α. By Witt's Lemma, we can assume that W has a basis

$\{e_1,\dots,e_\alpha\}$, where the vectors e_i are taken from a standard basis of V. Let $X = \langle e_{\alpha+1},\dots,e_m,f_{\alpha+1},\dots,f_m,a\rangle$ and $Y = \langle f_1,\dots,f_\alpha\rangle$. Then $V = (W \bigoplus Y) \perp X$, $W^\perp = W \perp X$. Finally let $U = C_{I(V)}(W, W^\perp/W, V/W^\perp)$, and $N = N_{I(V)}(W,Y,X)$. The relation among these groups is given in the following lemmas.

Lemma 5.4. *Let $(V,\mathbb{F}_q,Q)$ be a classical orthogonal geometry with $sgn(Q) = +$. Assume that $dim(V) = 2m$ and $\beta = \{e_1,\dots,e_m,f_1,\dots,f_m\}$ is a standard basis of V. Let $W_1 = \langle e_1,\dots,e_m\rangle, W_2 = \langle f_1,\dots,f_m\rangle$, and $T_0 = N_{I(V)}(W_1,W_2)$, where $V = \langle\beta\rangle$. Then*
(i) $T_0 \cong GL_m(q)$ and T_0 acts naturally on W_1.
(ii) As T_0-modules we have $W_2 \cong W_1^*$.
(iii) $T_0 \cap \Omega(V) = \{x \in T_0 \mid det_{W_1}(x) \in (\mathbb{F}_q^*)^2\}$.

Proof. This is a special case of [21, Lemma 4.1.9]). □

The next lemma is also a special case of [21, Lemma 4.1.12].

Lemma 5.5. *Let W be a totally singular subspace of V. Keeping the notation above, we have:*
(i) $M_I = U : N$;
(ii) $N = T_0 \times I(X)$, where $GL_\alpha(q) \simeq T_0 \leqslant I(W \oplus Y)$, and T_0 acts naturally on W; and as T_0-modules we have $Y \cong W$;
(iii) U is a p-group and $U \leqslant \Omega(V)$.

It follows from (i) and (iii) of Lemma 5.5 that $M_\Omega = U(N \cap \Omega(V))$.

Proposition 5.6. *Assume M is of type P_α. Then M has at most two orbits in $\mathfrak{E}_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7 and so M is in Table 1.*

Proof. We will show that M_Ω has at most two orbits on $\mathfrak{E}_\xi(V)$, and hence we deduce that M also has at most two orbits on $\mathfrak{E}_\xi(V)$. Assume the notation above and denote by f the associated bilinear form of Q. From definition M stabilizes a totally singular subspace W of dimension α. Let Y and X be defined as above. By [21, Proposition 2.6.1], X has a basis $\beta_X = \{x_1,\dots,x_s\}$, with $s = 2m+1-2\alpha = 2(m-\alpha)+1$, such that $[f_{\downarrow X}]_{\beta_X} = \lambda\mathbf{I}_s$, where $D(X) \equiv \lambda(\text{mod } (\mathbb{F}^*)^2)$. Let $\beta = \{e_1,\dots,e_\alpha,x_1,\dots,x_s,f_1,\dots,f_\alpha\}$. Let $x \in X$ be a non-singular point with $\xi = \rho_X(x)$. As $sgn(W \oplus Y) = +$, $\xi = \rho_V(x)$. If $\alpha < m$, then $s = dim(X) = 2(m-\alpha)+1 \geqslant 3$, so that X contains both plus and minus points. Otherwise, X has no minus points. As $\Omega(X) \leqslant M_\Omega$, and $U \leqslant M_\Omega$, we have $x(\Omega(X)U) \subseteq xM_\Omega$. For any $v \in x\Omega(X)$, $w \in W$, we will show that there exists $u \in U$ such that $vu = v+w$, which implies that $|\langle x\rangle\Omega(X)U| = |\mathfrak{E}_\xi(X)||W|$. Thus $|\langle x\rangle M_\Omega| \geqslant \frac{1}{2}|x\Omega(X)U| = |W||\mathfrak{E}_\xi(X)| = 3^\alpha\frac{1}{2}3^{m-\alpha}(3^{m-\alpha}+\xi 1)) = \dfrac{1}{2}3^m(3^{m-\alpha}+\xi 1)$. Therefore

$$|\langle x\rangle M_\Omega| \geqslant \frac{1}{2}3^m(3^{m-\alpha}+\xi 1). \tag{5.5}$$

Let $\widehat{B} = [f]_\beta$. Then

$$\widehat{B} = \begin{pmatrix} 0 & 0 & I_\alpha \\ 0 & \lambda I_s & 0 \\ I_\alpha & 0 & 0 \end{pmatrix}.$$

We have $[v]_\beta = (0,a,0)$ and $[w]_\beta = (b,0,0)$, where a,b are row vectors in $\mathbb{F}^s$ and $\mathbb{F}^\alpha$, respectively. Since v is non-singular, v is non-zero. Choose $B \in M_{s\times\alpha}(\mathbb{F})$ such that $aB = b$. Let $C = -\lambda^{-1}B^t, A + A^t = -\lambda^{-1}B^tB = -\lambda CC^t$, and

$$u = \begin{pmatrix} I_\alpha & 0 & 0 \\ B & I_s & 0 \\ A & C & I_\alpha \end{pmatrix}.$$

Then $u\widehat{B}u^t = \widehat{B}$, u centralizes $W, W^\perp/W$ and $V/W^\perp$. Thus $u \in U$ and $vu = v + w$.

Let $z = \eta e_1 + f_1 \in V$, where $\eta = Q(x)$. Then $Q(z) = Q(x)$ so that $\langle x \rangle$ and $\langle z \rangle$ belong to the same Ω-orbit of non-singular points in V. Let $T = \frac{1}{2}T_0$. By Lemma 5.4, $SL_\alpha(3) \simeq T \leqslant N \cap \Omega \leqslant M_\Omega$, and so $TU \leqslant M_\Omega$, where T_0, N are as in Lemma 5.5. Thus $z(TU) \subseteq zM_\Omega$. We find the stabilizer $(TU)_z$ in TU of vector z. For any $g \in TU$, there exist $h \in T, u \in U$ such that $g = hu$. We have

$$[h]_\beta = \begin{pmatrix} D & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & D^* \end{pmatrix}, \text{ and } [u]_\beta = \begin{pmatrix} I_\alpha & 0 & 0 \\ B & I_s & 0 \\ A & C & I_\alpha \end{pmatrix},$$

where $D = (d_{ij}) \in SL_\alpha(3)$, D^* its inverse transpose, and $C = -\lambda^{-1}B^t, A + A^t = -\lambda^{-1}B^tB = -\lambda CC^t$. Thus $[g]_\beta = [hu]_\beta = [h]_\beta[u]_\beta$. Suppose $g \in (TU)_z$. Write $E_1 = (\underbrace{\eta 1, 0, \ldots, 0}_{\alpha})$ and $F_1 = (\underbrace{1, 0, \ldots, 0}_{\alpha})$. Then $[z]_\beta = (E_1, 0, F_1)$. As $zg = z$, we have

$$(E_1, 0, F_1) \begin{pmatrix} D & 0 & 0 \\ B & I_s & 0 \\ D^*A & D^*C & D^* \end{pmatrix} = (E_1, 0, F_1),$$

or $(E_1D + F_1D^*A, F_1D^*C, F_1D^*) = (E_1, 0, F_1)$, hence

$$\begin{cases} F_1D^* & = & F_1 \\ F_1D^*C & = & 0 \\ E_1D + F_1D^*A & = & E_1 \end{cases}$$

Since $F_1D^* = F_1$,

$$D^* = \begin{pmatrix} 1 & 0 \\ b & D_1^* \end{pmatrix}, D = \begin{pmatrix} 1 & -b^tD_1 \\ 0 & D_1 \end{pmatrix}, \text{ and } D^{-1} = \begin{pmatrix} 1 & b^t \\ 0 & D_1^{-1} \end{pmatrix},$$

where $D_1 \in SL_{\alpha-1}(3)$ and b is a column vector of size $\alpha - 1$. As $F_1D^*C = 0$ and $F_1D^* = F_1$, we have $F_1C = 0$. Hence

$$C = \begin{pmatrix} 0 & 0 \\ c_0 & C_1 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a \\ a_0 & A_1 \end{pmatrix},$$

where $C_1 \in M_{\alpha-1,s-1}(3)$, c_0 is a column vector of size $\alpha-1$, $A_1 \in M_{\alpha-1}(3), a_{11} \in \mathbb{F}$ and a, a_0 are row, column vectors, respectively, of size $\alpha-1$. Now as $E_1D+F_1D^*A = E_1D + F_1A = E_1$, we have

$$(\xi, 0) \begin{pmatrix} 1 & -b^tD_1 \\ 0 & D_1 \end{pmatrix} + (1, 0) \begin{pmatrix} a_{11} & a \\ a_0 & A_1 \end{pmatrix} = (\eta 1, 0).$$

It follows that $(\eta 1 + a_{11}, a - \eta b^tD_1) = (\eta 1, 0)$. Therefore $a_{11} = 0$ and $a = \eta b^tD_1$. Finally as $A + A^t = -\lambda CC^t$, we have

$$\begin{pmatrix} 0 & a_0^t + \eta b^tD_1 \\ a_0 + \eta D_1^tb & A_1 + A_1^t \end{pmatrix} = -\lambda^{-1} \begin{pmatrix} 0 & 0 \\ 0 & c_0c_0^t + C_1C_1^t \end{pmatrix},$$

hence $a_0 = -\xi D_1^t b$, $A_1 + A_1^t = -\lambda(c_0c_0^t + C_1C_1^t)$ and

$$A = \begin{pmatrix} 0 & \eta b^t D_1 \\ -\eta D_1^t b & A_1 \end{pmatrix}.$$

In summary, for any $g \in (TU)_z$, we have

$$[g]_\beta = \begin{pmatrix} D & 0 & 0 \\ B & I_s & 0 \\ D^*A & D^*C & D^* \end{pmatrix} = \begin{pmatrix} D & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & D^* \end{pmatrix} \begin{pmatrix} I_\alpha & 0 & 0 \\ B & I_s & 0 \\ A & C & I_\alpha \end{pmatrix},$$

where

$$D = \begin{pmatrix} 1 & -b^t D_1 \\ 0 & D_1 \end{pmatrix}, D^* = \begin{pmatrix} 1 & 0 \\ b & D_1^* \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ c_0 & C_1 \end{pmatrix}, A = \begin{pmatrix} 0 & \xi b^t D_1 \\ -\xi D_1^t b & A_1 \end{pmatrix},$$

$B = -\lambda C^t \in M_{s,\alpha}(3)$, with $C_1 \in M_{\alpha-1,s-1}(3)$, $A_1 \in M_{\alpha-1}(3), b, c_0 \in M_{\alpha-1,1}(3)$, $A_1 + A_1^t = -\lambda^{-1}(c_0c_0^t + C_1C_1^t)$, $A \in M_\alpha(3), D, D^* \in SL_\alpha(3), C \in M_{\alpha,s}(3)$.

We see that the subgroup of $SL_\alpha(3)$ generated by all matrices D is isomorphic to $U_0 : SL_{\alpha-1}(3)$, where U_0 is an elementary abelian subgroup of order $3^{\alpha-1}$. Given such D, there are $3^{\alpha-1+(\alpha-1)(s-1)}$ choices for C and $3^{\frac{1}{2}(\alpha-1)(\alpha-2)}$ choices for A. Therefore

$$|(TU)_z| = 3^{\alpha-1}|SL_{\alpha-1}(3)|3^{\alpha-1+(\alpha-1)(s-1)+\frac{1}{2}(\alpha-1)(\alpha-2)}.$$

Since $|U| = 3^{\alpha s+\frac{1}{2}\alpha(\alpha-1)}$, we have $|z(TU)| = |TU : (TU)_z| = 3^{s+\alpha-1}(3^\alpha - 1)$. Thus

$$|\langle z\rangle M_\Omega| \geqslant \frac{1}{2}3^{s+\alpha-1}(3^\alpha - 1). \tag{5.6}$$

It follows from (5.5) and (5.6) that $|\mathfrak{E}_\xi(V)| \geqslant |\langle x\rangle M_\Omega| + |\langle z\rangle M_\Omega| \geqslant \dfrac{1}{2}3^m(3^{m-\alpha} + \xi1) + \frac{1}{2}3^{2m-\alpha}(3^\alpha - 1) = \frac{1}{2}3^m(3^m + \xi1) = |\mathfrak{E}_\xi(V)|$. Therefore $|\langle x\rangle M_\Omega| = |\langle x\rangle\Omega(X)U|$, $|\langle z\rangle M_\Omega| = |\langle z\rangle M_\Omega U|$, so that $\mathfrak{E}_\xi(V) = \langle x\rangle M_\Omega \cup \langle z\rangle M_\Omega$. Hence M_Ω has at most two orbits on $\mathfrak{E}_\xi(V)$. Clearly as $M_\Omega \leqslant M$, $\langle x\rangle M_\Omega \subseteq \langle x\rangle M$ and similarly $\langle z\rangle M_\Omega \subseteq \langle z\rangle M$. Since $\mathfrak{E}_\xi(V) = \langle x\rangle M_\Omega \cup \langle z\rangle M_\Omega$ and $\langle x\rangle M \cup \langle z\rangle M \subseteq \mathfrak{E}_\xi(V)$, it implies that $\mathfrak{E}_\xi(V) = \langle x\rangle M \cup \langle z\rangle M$. Thus M has at most two orbits on $\mathfrak{E}_\xi(V)$. □

5.2.2. *The imprimitive subgroups $\mathcal{C}_2$.* Let V be a vector space over a finite field $\mathbb{F}_q$ with $n = dimV$. A *subspace decomposition* $\mathcal{D} = \{V_1, \dots, V_b\}$ of V is a set of subspaces $V_1, \dots, V_b$ of V with $b \geqslant 2$ such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_b$. Let $\mathfrak{G}$ be a subgroup of $GL(V)$. The *stabilizer* in $\mathfrak{G}$ of $\mathcal{D}$ is the group $N_{\mathfrak{G}}\{V_1, \dots, V_b\}$, which is the subgroup of $\mathfrak{G}$, permuting the spaces V_i amongst themselves and denoted by $\mathfrak{G}_{\mathcal{D}}$. The *centralizer in* $\mathfrak{G}$ of $\mathcal{D}$, is the group $\mathfrak{G}_{(\mathcal{D})} = N_{\mathfrak{G}}(V_1, \dots, V_b)$, which is a subgroup of $\mathfrak{G}$ fixing each V_i. We also define $\mathfrak{G}^{\mathcal{D}} = \mathfrak{G}_{\mathcal{D}}/\mathfrak{G}_{(\mathcal{D})}$. If the spaces V_i in the subspace decomposition $\mathcal{D}$ all have the same dimension α, then $\mathcal{D}$ is called an *α-decomposition.* If the $V_i's$ are non-degenerate and pairwise orthogonal, then $\mathcal{D}$ is said to be *non-degenerate.* For any vector $v \in V$, v can be written in the form $v = v_1 + v_2 + \cdots + v_b$, where $v_i \in V_i$. We define the *$\mathcal{D} - length$* of v to be the number of non-zero vectors v_i appearing in v, and denote by $\mathcal{D}_b^k$, the number of all points of $\mathcal{D}-$length $k, 1 \leqslant k \leqslant b$. The members of $\mathcal{C}_2(\Xi)$ are the stabilizers in Ξ of α-decomposition $\mathcal{D}$ of V such that $\mathcal{D}$ is non-degenerate and if $\alpha = 1$ then $q = p$, a prime.

Lemma 5.7. *Let M_Ω be the stabilizer in $\Omega(V)$ of a non-degenerate α-decomposition $\mathcal{D}$ and $n = b\alpha$. Then*

(i) If $\alpha > 1$*, then* $M_\Omega \cong \Omega(V)_{(\mathcal{D})} S_b$.
(ii) If $\alpha = 1$*, and* $q \equiv \pm 3 \pmod 8$*, then* $M_\Omega \cong \Omega(V)_{(\mathcal{D})} A_n$.

Proof. This is Propositions 4.2.14 and 4.2.15 in [21]. □

Lemma 5.8. *Assume* $\mathcal{D}$ *is a* 1*-decomposition of* V *with* q *odd. For* $1 \leqslant k \leqslant n$, $|\mathcal{D}_n^k| = (q-1)^{k-1}\binom{n}{k}$.

Proof. Without loss of generality, we can assume V has an orthonormal basis $\beta = \{v_1, \dots, v_n\}$. If $v \in V$ has $\mathcal{D}$-length k then v is a linear combination of a set of k basic vectors taken from the basis β, with coefficients in $\mathbb{F}_q^*$. Clearly, there are $\binom{n}{k}$ choices for k-sets, and for each k-set, there are $(q-1)^{k-1}$ points of length k. The result follows. □

Lemma 5.9. *Assume* G *is nearly simple primitive rank* 3 *of type* $\Omega_n(3)$ *and* M *is of type* $O_1(3) \wr S_n$ *with* $n = 2m+1$*. Let* $\beta = \{x_1, \dots, x_n\}$ *be an orthogonal basis for* V*.*
(1) *If* $z = x_1$ *then* $d_1 = d_z = n-1$ *and* $c_1 = c_z = 0$*;*
(2) *If* $z = x_1 + x_2$ *then* $d_2 = d_z = n^2 - 5n + 7$ *and* $c_2 = c_z = 4n - 8$.

Proof. By multiplying a suitable non-zero constant to the quadratic form Q, we can assume that V has an orthonormal basis $\beta = \{x_1, \dots, x_n\}$. Setting $r_i = r_{x_i}$, the reflection along vector x_i. Then we have $I(V)_{(\mathcal{D})} = \langle r_i | 1 \leqslant i \leqslant n\rangle \cong 2^n$, and $\Omega(V)_{(\mathcal{D})} = \langle r_i r_j | 1 \leqslant i, j \leqslant n\rangle \cong 2^{n-1}$. For $1 \leqslant i \neq j \leqslant n$, we see that $r_{x_i - x_j}$ permutes $\{x_i, x_j\}$, and fixes x_t for any $t \notin \{i, j\}$ and so $r_{x_i - x_j}$ acts as a transposition (i, j). Thus if we denote by J, the group generated by all reflections $r_{x_i - x_j}$, where $1 \leqslant i \neq j \leqslant n$, then $J \cong S_n$ and hence J_1, the subgroup of J generated by $r_{x_i - x_j} r_{x_r - x_s}$, with $i \neq j, r \neq s$ is isomorphic to A_n. By Lemma 5.7(*ii*), $M_\Omega = \Omega(V)_{(\mathcal{D})} J_1$. For any $1 \leqslant i \neq j \leqslant n$, as $(x_i, x_j) = 0$, r_j fixes x_i. Hence $\Omega(V)_{(\mathcal{D})}$ leaves invariant the point $\langle x_1\rangle$, and $\langle x_1 + x_2\rangle \Omega(V)_{(\mathcal{D})} = \{\langle x_1 + x_2\rangle, \langle x_1 - x_2\rangle\}$, as $(x_1 + x_2) r_2 r_3 = x_1 - x_2$. By [21, (4.2.17)], we have $M_I = M_\Omega \langle r_3, r_{x_3 - x_4}\rangle$. Thus $\langle x\rangle M_\Omega = \langle x\rangle M_I$ for any $x \in \{x_1, x_1 + x_2\}$, so that it suffices to compute the parameters for M_Ω in L. Since $n \geqslant 5$, A_n acts transitively on the set $\{1, 2, \dots, n\}$. Thus $\langle x_1\rangle M_\Omega = \{\langle x_1\rangle, \dots, \langle x_n\rangle\}$. Hence $1 + c_1 + d_1 = n$. Moreover as $(x_i, x_1) = 0$ for any $i > 1$, we have $d_1 = |x_1^\perp \cap \langle x_1\rangle M_\Omega| = |\{\langle x_2\rangle, \dots, \langle x_n\rangle\}| = n-1$, and so $c_1 = 0$. Similarly, as A_n acts doubly transitively on $\{1, \dots, n\}$, we have $\langle x_1 + x_2\rangle M_\Omega = \langle x_1 + x_2\rangle \Omega(V)_{(\mathcal{D})} J_1 = \{\langle x_1 + x_2\rangle, \langle x_1 - x_2\rangle\} J_1$. Thus by Lemma 5.8 $1 + c_2 + d_2 = |\langle x_1 + x_2\rangle M_\Omega| = \mathcal{D}_n^2 = n(n-1)$. For any $\langle v\rangle \in \langle x_1 + x_2\rangle^\perp \cap \langle x_1 + x_2\rangle M_\Omega$, $\langle v\rangle = \langle x_i \pm x_j\rangle$ for some $i \neq j \in \{1, \dots, n\}$ and $(v, x_1 + x_2) = 0$. Clearly $\langle v\rangle$ is generated by $x_1 - x_2$ or $x_i \pm x_j$ for some $i < j \in \{3, \dots, n\}$. By Lemma 5.8 again, $d_2 = |\langle x_1 + x_2\rangle^\perp \cap \langle x_1 + x_2\rangle M_\Omega| = \mathcal{D}_{n-2}^2 + 1 = n^2 - 5n + 7$, and so $c_2 = 4n - 8$. □

Proposition 5.10. *Assume* M *is of type* $O_1(3) \wr S_n$*, with* $n = 2m+1$*. There is an* M*-orbit on* $\mathfrak{E}_\xi(V)$ *such that equation (3.1) does not hold unless* $(n, \xi, r) = (5, +, t), (7, +, t)$ *or* $(5, -, s)$*, in which cases* M *has two orbits on* $\mathfrak{E}_\xi(V)$ *so that* $1_P^G \nleqslant 1_M^G$ *by Corollary 3.7, and hence* M *is in Table 1.*

Proof. Retain the notation in the previous lemma. By Propositions 2.5.10, and 2.5.13 in [21], $sgn(x_1^\perp) = (-)^m$ and $sgn(x_1 + x_2)^\perp = (-)^{m+1}$, as the discriminant of the corresponding subspaces is square or non-square respectively. When m is even x_1 is a plus vector and $x_1 + x_2$ is a minus vector and vice versa when m is odd. Let $x \in \{x_1, x_1 + x_2\}$. We consider the following cases:

(*i*) $\langle x\rangle$ is a plus point. If m is even then we choose $x = x_1$. By Lemma 5.9 $d = d_1 = n-1, c = c_1 = 0$. Then $k = -rd$, so r must be t and so $3^m - 1 = 4m$. This equation holds only when $m = 2$ and hence $n = 5$. If m is odd then choose $x = x_1+x_2$, and hence by Lemma 5.9 again $d = d_2 = (n-2)(n-3)+1, c = c_2 = 4n-8$. Then $c-2d = 14n-2n^2-22$. As $n \geqslant 5$, $2n^2+22 > 14n$ so that $c-2d < 0$. Therefore, equation (5.1) cannot hold. If equation (5.2) holds then $(3^{m-1}+1)(3^m-1+c-2d) = 2c$. It follows that $(3^{m-1} + 1)(3^m - 1 + 14n - 2n^2 - 22) = 8(2m - 1)$. If $m \geqslant 5$, then $3^{m-1} + 1 > 8(2m - 1)$, hence this equation cannot hold. For $2 \leqslant m \leqslant 4$, the equation occurs only when $m = 3$. Thus equation (3.1) holds only when $r = t$ and $m = 3$ or $n = 7$.

(*ii*) $\langle x\rangle$ is a minus point. If m is odd then $x = x_1$ and $d = n - 1, c = 0$. Then $k = -rd$. It follows that $r = t$, and hence $3^m + 1 = 4m$. Since $m \geqslant 2, 3^m + 1 > 4m$, so that this equation cannot hold. If m is even, then $x = x_1 + x_2$ and $d = (n - 2)(n - 3) + 1, c = 4n - 8$. We have $2d - c = 2n^2 + 22 - 14n$. If equation (5.2) holds then $(3^{m-1} - 1)(3^m + 1 + 2d - c) = 2c$, hence $(3^{m-1} - 1)(3^m + 1 + 2n^2 + 22 - 14n) = 8(2m - 1)$. Since $2n^2 + 22 - 14n > 0$, for any $m \geqslant 3$, $(3^{m-1} - 1)(3^m + 1 + 2n^2 + 22 - 14n) > (3^{m-1} - 1)(3^m + 1) > 8(2m - 1)$; when $m = 2$, $(3^{m-1}-1)(3^m+1+2n^2+22-14n) = 24 = 8(2m-1)$. If equation (5.1) holds then $2d - c = 2n^2 + 22 - 14n = 3^m + 1$. We can check that $3^m + 1 > 2n^2 + 22 - 14n$ for any $m \geqslant 2$. Thus equation (3.1) holds only when $m = 2$ or $n = 5$ and $r = s$.

To finish the proof, we need to verify that when these cases happen then equation (3.1) also holds for all $\langle x\rangle \in \mathfrak{E}_\xi(V)$. In view of Corollary 3.7, we will show that there are only two orbits of non-singular points of specified types. Firstly, suppose that $n = 5$. Then $m = 2$, and $|\mathfrak{E}_\xi(V)| = \frac{1}{2}3^m(3^m + \xi 1)$. In this case, $\langle x_1\rangle M_\Omega$ and $\langle x_1+x_2+x_3+x_4\rangle M_\Omega$ are two orbits of plus points with orbit sizes 5 and $2^3\binom{5}{4} = 40$, respectively. As $|\mathfrak{E}_+(V)| = \frac{1}{2}3^2(3^2 + 1) = 45 = 5 + 40$, there are only two orbits of plus points. Similarly, $\langle x_1 + x_2\rangle M_\Omega$ and $\langle x_1 + x_2 + x_3 + x_4 + x_5\rangle M_\Omega$ are two orbits of minus points with orbit sizes $2\binom{5}{2} = 20$ and $2^4\binom{5}{5} = 16$, respectively. Since $|\mathfrak{E}_-(V)| = \frac{1}{2}3^2(3^2-1) = 36 = 20+16$, there are exactly two orbits of minus points. Finally, suppose that $n = 7$, and $\xi = +$. Then $m = 3$ and $|\mathfrak{E}_+(V)| = \frac{1}{2}3^3(3^3 + 1) = 378$. In this case, $\langle x_1 + x_2\rangle M_\Omega$ and $\langle x_1 + x_2 + x_3 + x_4 + x_5\rangle M_\Omega$ are two orbits of plus points with orbit sizes $2\binom{7}{2} = 42$ and $2^4\binom{7}{5} = 336$. Since $336 + 42 = 378$, there are only two orbits of plus points. This completes the proof. □

We next consider the case when $\alpha > 1$. Since $\dim V$ is odd, it follows that α and b are both odd. Write $\alpha = 2a + 1, b = 2b_1 + 1$.

Proposition 5.11. *Assume M is of type $O_\alpha(3) \wr S_b$, with $\alpha > 1$ odd. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1-3.*

Proof. We can assume that V has an orthonormal basis which is the union of orthonormal bases of all V_i. Let $N = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_b \leqslant M_\Omega$. By [21, Lemma 4.2.8], we have $M_I = I_1 \wr S_b$. Thus $\Omega_1 \wr S_b \leqslant M_\Omega \leqslant M_I$. Since $\alpha > 1$ is odd, $\alpha \geqslant 3$ and so V_1 contains both plus and minus points. Let $x_\xi \in V_1$ be a non-singular vector of type $\xi \in \{\pm\}$. Clearly $\langle x_\xi\rangle\Omega_1 \wr S_b = \langle x_\xi\rangle I_1 \wr S_b$, we conclude that $\langle x_\xi\rangle M_\Omega = \langle x_\xi\rangle M_I = \langle x_\xi\rangle NS_b$. Thus we only need to compute the parameter for M_Ω in L. Since $S_b \leqslant M_\Omega$ permutes the $V_i's$, and Ω_i centralizes V_1, for all $i > 1$, $\langle x_\xi\rangle M_\Omega = \langle x_\xi\rangle NS_b = \langle x_\xi\rangle\Omega_1 S_b = \mathfrak{E}_\xi(V_1)S_b = \cup_{i=1}^b \mathfrak{E}_\xi(V_i)$. Thus $A = |\langle x_\xi\rangle M_\Omega| = \frac{b}{2}3^a(3^a+\xi 1)$ by Lemma 5.1(i). Hence $A \leqslant \frac{1}{2}b \cdot 3^a(3^a + 1)$. In view of inequality (5.3), it suffices to show that $\frac{1}{2}3^m \geqslant \frac{1}{2}b \cdot 3^a(3^a+1)$. Since $m = ba + b_1$, this inequality is equivalent to $3^{ba+b_1} \geqslant b3^a(3^a+1)$. As $b \geqslant 3$ and $a \geqslant 1$, $3^{ba} \geqslant 3^{3a} = 3^a \cdot 3^{2a} \geqslant 3 \cdot 3^{2a} > 3^{2a} + 3^a$. If we can prove that $3^{b_1} = 3^{\frac{b-1}{2}} \geqslant b$, then clearly $3^{ba} \cdot 3^{b_1} \geqslant b3^a(3^a+1)$, and we are done. To show that $3^{b_1} \geqslant b$, we will argue by induction on $b \geqslant 3$. When $b = 3$ then $3^{\frac{b-1}{2}} = 3 \geqslant 3 = b$. Suppose that $3^{\frac{b-1}{2}} \geqslant b$. Then $3^{\frac{(b+1)-1}{2}} = 3^{\frac{b-1}{2}}\sqrt{3} \geqslant b\sqrt{3}$, by induction assumption. We have $3b^2 = b^2 + 2b^2 \geqslant b^2 + 6b > b^2 + 2b + 1 = (b+1)^2$, as $b \geqslant 3$. Thus $b\sqrt{3} \geqslant b+1$. Hence $3^{\frac{b+1-1}{2}} \geqslant b+1$. The result follows. □

5.2.3. *The field extension subgroups $\mathcal{C}_3$.* Let $\mathbb{F}_\sharp$ be a field extension of $\mathbb{F} = \mathbb{F}_3$ of degree α, where α is a prime divisor of $n = dimV$. Then V acquires the structure of an $\mathbb{F}_\sharp$-vector space in a natural way. Write $V_\sharp$ for V regarded as a vector space over $\mathbb{F}_\sharp$. Denote by T the trace map from $\mathbb{F}_\sharp$ to $\mathbb{F}$. If $Q_\sharp$ is a quadratic form on $(V_\sharp, \mathbb{F}_\sharp)$ then $Q = TQ_\sharp$ is a quadratic form on $(V, \mathbb{F})$. Write $f_\sharp$ for the associated bilinear form of $Q_\sharp$. Denote by $N = N_{\mathbb{F}_\sharp/\mathbb{F}}$ the norm map of $\mathbb{F}_\sharp$ over $\mathbb{F}$. Let μ, ν be the generators for $\mathbb{F}_\sharp^*$ and $Gal(\mathbb{F}_\sharp/\mathbb{F})$, respectively. Also the trace map from $\mathbb{F}_\sharp$ to $\mathbb{F}$ defines a non-degenerate bilinear form on $\mathbb{F}_\sharp$. Let $Q_T : \mathbb{F}_\sharp \to \mathbb{F}$ be a map defined by $Q_T(x) = -T(x^2)$ for $x \in \mathbb{F}_\sharp$. Then Q_T is a quadratic form on $\mathbb{F}_\sharp$ and $f_T(x,y) = T(xy)$ is the bilinear form associated to Q_T. Then $(\mathbb{F}_\sharp, Q_T, \mathbb{F})$ is an orthogonal geometry.

Lemma 5.12. *Let $\beta_T = \{\zeta_1, \zeta_2, \ldots, \zeta_\alpha\}$ be an $\mathbb{F}$-basis of $\mathbb{F}_\sharp = \mathbb{F}_{p^\alpha}$, where $p = 3$, and $v_\sharp \in V_\sharp$ be such that $f_\sharp(v_\sharp, v_\sharp) = \lambda \in \mathbb{F}_\sharp^*$. Then*
(i) $D(\mathbb{F}_\sharp) \equiv det(f_{\beta_T}) \equiv (-1)^{\alpha-1} (mod\ (\mathbb{F}^*)^2)$;
(ii) $span_{\mathbb{F}_\sharp}(v_\sharp)$ is a non-degenerate α-subspace in V with discriminant $D(\mathbb{F}_\sharp)N(\lambda)$.

Proof. From definition, we have

$$f_{\beta_T} = \begin{pmatrix} T(\zeta_1^2) & T(\zeta_1\zeta_2) & \ldots & T(\zeta_1\zeta_\alpha) \\ T(\zeta_2\zeta_1) & T(\zeta_2^2) & \ldots & T(\zeta_2\zeta_\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ T(\zeta_\alpha\zeta_1) & T(\zeta_\alpha\zeta_2) & \ldots & T(\zeta_\alpha^2) \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_\alpha \\ \zeta_1^p & \zeta_2^p & \cdots & \zeta_\alpha^p \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{p^{\alpha-1}} & \zeta_2^{p^{\alpha-1}} & \ldots & \zeta_\alpha^{p^{\alpha-1}} \end{pmatrix}$$

and $E = diag(\lambda, \lambda^p, \ldots, \lambda^{p^{\alpha-1}})$. As $T(a) = \sum_{i=0}^{\alpha-1} a^{p^i}$ for any $a \in \mathbb{F}_\sharp$, $X^tX = f_{\beta_T}$, hence $det(f_{\beta_T}) = det(X^tX) = det(X)^2$. Since $det(f_{\beta_T}) \in \mathbb{F}^*$ and $detX \in \mathbb{F}_\sharp$, if α is odd then clearly $detX \in \mathbb{F}^*$, as $\mathbb{F}_\sharp$ does not have any subfields of degree 2 over $\mathbb{F}$. Thus $(detX)^2 \in (\mathbb{F}^*)^2 = \{1\}$, so $det(f_{\beta_T}) = (detX)^2 = 1$. Now suppose that $\alpha = 2$. Let ζ be a root of $x^2 - x - 1$ in $\overline{\mathbb{F}}$, and let $\tau = \zeta + 1$. Then $\tau^2 = -1, \mathbb{F}_\sharp = \mathbb{F}(\tau)$ and $T(\tau) = 0$. Choose $\beta_T = \{1, \tau\}$. Then

$$f_{\beta_T} = \begin{pmatrix} T(1) & T(\tau) \\ T(\tau) & T(\tau^2) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Hence $det(f_{\beta_T}) = -1$. This proves (i). Let $\beta = \{\zeta_1 v_\sharp, \zeta_2 v_\sharp, \dots, \zeta_\alpha v_\sharp\}$ and $W = \langle v_\sharp\rangle_{\mathbb{F}_\sharp}$. As $(\zeta_i v_\sharp, \zeta_j v_\sharp) = T(f_\sharp(\zeta_i v_\sharp, \zeta_j v_\sharp)) = T(\zeta_i\zeta_j f_\sharp(v_\sharp, v_\sharp)) = T(\lambda\zeta_i\zeta_j)$, we have

$$f_\beta = \begin{pmatrix} T(\lambda\zeta_1^2) & T(\lambda\zeta_1\zeta_2) & \dots & T(\lambda\zeta_1\zeta_\alpha) \\ T(\lambda\zeta_2\zeta_1) & T(\lambda\zeta_2^2) & \dots & T(\lambda\zeta_2\zeta_\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ T(\lambda\zeta_\alpha\zeta_1) & T(\lambda\zeta_\alpha\zeta_2) & \dots & T(\lambda\zeta_\alpha^2) \end{pmatrix}.$$

Obviously $X^t E X = f_\beta$. Therefore $det(f_\beta) = det(X)^2 N(\lambda) = D(\mathbb{F}_\sharp)N(\lambda)$. □

Proposition 5.13. *Assume M is of type $O_{\frac{n}{\alpha}}(3^\alpha)$ with $n = 2m+1$ and $\alpha \mid n$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $\alpha = 3$. In this case M has 3 orbits on $\mathfrak{E}_\xi(V)$ and equation (3.1) holds for all M-orbits on $\mathfrak{E}_\xi(V)$ with $r = s$, and hence M is in Table 1.*

Proof. Let $q = 3^\alpha$, and μ, ν be the generators for $\mathbb{F}_\sharp^*$ and $Gal(\mathbb{F}_\sharp/\mathbb{F})$, respectively. As n is odd, it follows that α is also odd. Write $\alpha = 2\alpha_1 + 1$ and $\frac{n}{\alpha} = 2b+1$. Then $m = b\alpha + \alpha_1$, where $n = 2m+1$. Multiplying by a suitable constant to the quadratic form $Q_\sharp$, we can assume that $D(Q_\sharp) = \square$. By [21, Proposition 2.6.1], there exists a basis $\beta_\sharp = \{w_1, w_2, \dots, w_{2b+1}\}$ of $(V_\sharp, Q_\sharp)$ such that $f_{\beta_\sharp} = I_{2b+1}$. Define $\phi_\sharp = \phi_{Q_\sharp,\beta_\sharp} = \phi_{\beta_\sharp}(\nu)$. Then $o(\phi_\sharp) = \alpha$. We will show that $\phi_\sharp \in \Omega$. Let $\beta_n = \{\zeta, \zeta^3, \dots, \zeta^{3^{\alpha-1}}\}$ be a normal basis of $\mathbb{F}_\sharp$ over $\mathbb{F}$, and $\beta_i = \beta_n \otimes w_i$, Since $(\zeta^{3^j} w_i)\phi_\sharp = \zeta^{3^{j+1}} w_i$, we obtain

$$(\phi_\sharp)_{\beta_i} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

As $det(\phi_\sharp)_{\beta_i} = (-1)^{\alpha-1} = 1$, $det(\phi_\sharp) = 1$. So $\phi_\sharp \in S$. As $[S:\Omega] = 2$ and $o(\phi_\sharp) = \alpha$ is odd, $\phi_\sharp \in \Omega$. Let $I = I(V, \mathbb{F}, Q)$, $I_\sharp = I(V_\sharp, \mathbb{F}_\sharp, Q_\sharp)$, and $\Omega_\sharp = \Omega(V_\sharp, \mathbb{F}_\sharp, Q_\sharp)$. Then by [21, (4.3.11)], $M_I = I_\sharp\langle\phi_\sharp\rangle \cong I_\sharp\mathbb{Z}_\alpha$. Since $\Omega_\sharp$ is perfect, $\Omega_\sharp \leqslant L$, hence $\Omega_\sharp \leqslant M_\Omega N_L(\mathbb{F}_\sharp) \leqslant M_I = I_\sharp\langle\phi_\sharp\rangle$. By [21, Proposition 4.3.17], $[M_\Omega : \Omega_\sharp] = \alpha$. As $\phi_\sharp \in \Omega \cap I_\sharp\langle\phi_\sharp\rangle$, $\phi_\sharp \in M_\Omega$, and hence $M_\Omega = \Omega_\sharp\langle\phi_\sharp\rangle$. It follows from [21, Lemma 2.10.5] that $\langle z\rangle\Omega_\sharp = \langle z\rangle I_\sharp$ for any non-singular point $\langle z\rangle$ in $(V_\sharp, \mathbb{F}_\sharp, Q_\sharp)$, so that $\langle z\rangle M_\Omega = \langle z\rangle M_I$, hence we only need to compute parameters for M_Ω in L. We first claim the following:

(1) $Q_\sharp(w\phi_\sharp) = Q_\sharp(w)^\nu$ for any $w \in V_\sharp$.

(2) $\langle z\rangle M_\Omega = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) \in \{\gamma, \gamma^\nu, \dots, \gamma^{\nu^{\alpha-1}}\}\}$, where $z \in V_\sharp$ with $\gamma = Q_\sharp(z)$.

For (1), assume that $w = \sum_{i=1}^{2b+1}\lambda_i w_i \in V_\sharp$. Then $w\phi_\sharp = \sum_{i=1}^{2b+1}(\lambda_i w_i)\phi_\sharp = \sum_{i=1}^{2b+1}\lambda_i^\nu w_i$, hence $Q_\sharp(w\phi_\sharp) = (\sum_{i=1}^{2b+1} -\lambda_i^2)^\nu = Q_\sharp(w)^\nu$. For (2), from (1) we have $Q_\sharp(z\phi_\sharp) = \gamma^\nu$, hence $\{Q_\sharp(z\phi_\sharp^j)\}_{j=1}^\alpha = \{\gamma, \gamma^\nu, \dots, \gamma^{\nu^{\alpha-1}}\}$. Thus $\langle z\rangle M_\Omega = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) \in \{\gamma, \gamma^\nu, \dots, \gamma^{\nu^{\alpha-1}}\}\}$. For a non-zero vector $w \in V_\sharp$, consider $span_{\mathbb{F}_\sharp}(w)$ as an α-subspace in V.

(a) **Case** $\alpha > 3$. Let $z \in \{w_1, w_1 + w_2\}$. Then $Q_\sharp(z) = \mp 1$ and $Q(z) = TQ_\sharp(z) = \mp\alpha \neq 0$ so z is non-singular in V. Also as $Q_\sharp(z) = \mp 1$ is fixed under ν, by (2) we have $\langle z\rangle M_\Omega = \langle z\rangle\Omega_\sharp$, and by Lemma 2.2, $|\langle z\rangle M_\Omega| = |\langle z\rangle\Omega_\sharp| = \frac{1}{2}(q^{2b} + \varepsilon q^b)$, with $\varepsilon = sgn(z^{\perp}_{V_\sharp})$. We have $\langle z\rangle M_\Omega \cap z^\perp = \langle z\rangle \Omega_\sharp \cap z^\perp = \{v \in V_\sharp \mid Q_\sharp(v) = Q_\sharp(z), Tf_\sharp(v,z) = 0\}$. For $w \in \langle z\rangle M_\Omega \cap z^{\perp}_{V}$, write $w = \varphi f_\sharp(z,z)^{-1}z + w_0$, where $w_0 \in z^{\perp}_{V_\sharp}$, and $T(\varphi) = 0$. Then $f_\sharp(w,z) = \varphi$ and $Q_\sharp(w_0) = Q_\sharp(z)^{-1}(Q_\sharp(z)^2 - \varphi^2)$. As $T(\pm Q_\sharp(z)) = T(\pm 1) \neq 0$, $Q_\sharp(w_0) \neq 0$ for any $\varphi \in \mathbb{F}_\sharp$ with $T(\varphi) = 0$. When $\varphi \in \ker T$ is fixed, as $dim_{\mathbb{F}_\sharp}(z^{\perp}_{V_\sharp}) = 2b$ and $sgn(z^{\perp}_{V_\sharp}) = \varepsilon$, by Lemma 2.2, there are $q^{2b-1} - \varepsilon q^{b-1}$ vectors w_0 with $Q(w_0) = Q_\sharp(z)^{-1}(1-\varphi^2) \neq 0$. Also $dim_{\mathbb{F}}(\mathrm{Ker}T) = \alpha - 1$, we conclude that $d_z = |\langle z\rangle M_\Omega \cap z^{\perp}_{V}| = \frac{1}{2}3^{\alpha-1}(q^{2b-1} - \varepsilon q^{b-1}) = \frac{1}{6}(q^{2b} - \varepsilon q^b)$. Thus $c_z = \frac{1}{3}(q^{2b} + \varepsilon 2q^b) - 1 = (\varepsilon 3^{b\alpha} - 1)(\varepsilon 3^{b\alpha - 1} + 1)$, and $c_z - 2d_z = \varepsilon q^b - 1 = \varepsilon 3^{b\alpha} - 1$. Assume (5.1) holds. Then $c_z - 2d_z = \varepsilon 3^{b\alpha} - 1 = \xi 3^{m-1} - 1$, where $\xi = sgn(z^{\perp}_{V})$. The latter equation yields $m = b\alpha + 1$. Recall that $m = b\alpha + \alpha_1$, hence $\alpha_1 = \frac{1}{2}(\alpha - 1) = 1$, this forces $\alpha = 3$, a contradiction. Suppose (5.2) holds. Then $(3^{m-1} + \xi 1)(3^m - \xi 1 + \xi(c - 2d)) = 2c$, hence $(3^{m-1} + \xi 1)(3^m - \varepsilon\xi 3^{b\alpha} - \xi 2) = 2(3^{b\alpha} - \varepsilon 1)(3^{b\alpha - 1} + \varepsilon 1)$. We will show that $3^{m-1} + \xi > 2(3^{b\alpha} - \varepsilon 1)$ and $3^m - \varepsilon\xi 3^{b\alpha} - \xi 2 > 3^{b\alpha - 1} + \varepsilon 1$, so that after multiplying these two inequalities side by side, we get a contradiction. For the first inequality, we have $3^{m-1} + \xi \geqslant 3^{b\alpha + \alpha_1 - 1} - 1 \geqslant 3^{\alpha_1 - 1}3^{b\alpha} - 1 \geqslant 3 \cdot 3^{b\alpha} - 1$, as $\alpha_1 \geqslant 2$. Now $2(3^{b\alpha} - \varepsilon 1) \leqslant 2(3^{b\alpha} + 1)$. It suffices to show that $3 \cdot 3^{b\alpha} - 1 > 2(3^{b\alpha} + 1)$. This inequality is equivalent to $3^{b\alpha} > 3$. This is true because $b\alpha > 1$. For the second inequality, as $3^{b\alpha} - 3 > 0$, we have $3^m - \varepsilon\xi 3^{b\alpha} - \xi 2 \geqslant 3^{b\alpha + \alpha_1} - 3^{b\alpha} - 2 = (3^{\alpha_1} - 1)3^{b\alpha} - 2 > 2 \cdot 3^{b\alpha} - 2 = (3^{b\alpha} + 1) + (3^{b\alpha} - 3) \geqslant 3^{b\alpha - 1} + \varepsilon 1$.

(*b*) **Case** $\alpha = 3$. Let ω be a root of $x^3 - x + 1$ in $\overline{\mathbb{F}}$. Then $\langle \omega \rangle = \mathbb{F}^*_\sharp$, and $KerT$ has a basis $\{1, \omega\}$ with $T(\omega^2) = -1$. We have $m = 3b + 1, q = 3^3$. Let $x_1 = \omega w_1, x_2 = \omega^2 w_1, x_3 = \omega^4(w_1 + w_2)$ and $y_1 = \omega(w_1 + w_2), y_2 = \omega^2(w_1 + w_2), y_3 = \omega^4 w_1$. For $i = 1, \ldots, 3$, we have $Q_\sharp(x_i) \neq 0, Q_\sharp(y_i) \neq 0$, and $Q(x_i) = 1, Q(y_i) = -1$ and so x_i, y_i are non-singular in both $V_\sharp$ and V. Also all $x_i's$ $(y_i's)$ belong to different $\Omega_\sharp$-orbits but they are in the same Ω-orbits. For each $i = 1, 2$, we have $x_i{}^{\perp}_{V_\sharp} = \langle w_2, \ldots, w_{2b+1}\rangle$ and $y_i{}^{\perp}_{V_\sharp} = \langle w_1 - w_2, w_3, \ldots, w_{2b+1}\rangle$, so that $D(x_i{}^{\perp}_{V_\sharp}) = \square, D(y_i{}^{\perp}_{V_\sharp}) = \boxtimes$, and hence by [21, Proposition 2.5.10, 2.5.13], $sgn(x_i{}^{\perp}_{V_\sharp}) = (-)^b$ and $sgn(y_i{}^{\perp}_{V_\sharp}) = (-)^{b-1}$, where $dim x_i{}^{\perp}_{V_\sharp} = dim y_i{}^{\perp}_{V_\sharp} = 2b$. For $i = 3$, as computation above, we have $sgn(x_3{}^{\perp}_{V_\sharp}) = (-)^{b-1}$ and $sgn(y_3{}^{\perp}_{V_\sharp}) = (-)^b$. We now determine the type of x_i and y_i in $(V, \mathbb{F}, Q)$. Let $U = span_{\mathbb{F}_\sharp}(w_3) \perp \cdots \perp span_{\mathbb{F}_\sharp}(w_{2b+1}) \leqslant V$ be an $\mathbb{F}$-subspace. We have $x_1{}^{\perp}_{V} = \langle w_1, \omega^2 w_1\rangle \perp span_{\mathbb{F}_\sharp}(w_2) \perp U$, $x_2{}^{\perp}_{V} = \langle \omega w_1, (\omega^2 - \omega)w_1\rangle \perp span_{\mathbb{F}_\sharp}(w_2) \perp U$, $x_3{}^{\perp}_{V} = \langle (\omega + 1)(w_1 + w_2), (\omega^2 - 1)(w_1 + w_2)\rangle \perp span_{\mathbb{F}_\sharp}(w_1 - w_2) \perp U$, and similarly $y_1{}^{\perp}_{V} = \langle (w_1 + w_2), \omega^2(w_1 + w_2)\rangle \perp span_{\mathbb{F}_\sharp}(w_1 - w_2) \perp U$, $y_2{}^{\perp}_{V} = \langle \omega(w_1 + w_2), (\omega^2 - \omega)(w_1 + w_2)\rangle \perp span_{\mathbb{F}_\sharp}(w_1 - w_2) \perp U$, $y_3{}^{\perp}_{V} = \langle (\omega + 1)w_1, (\omega^2 - 1)w_1\rangle \perp span_{\mathbb{F}_\sharp}(w_2) \perp U$. By Lemma 5.12, $D(span_{\mathbb{F}_\sharp}(w_i)) = N(f_\sharp(w_i, w_i)) = N(1) = 1 = \square$ and $D(span_{\mathbb{F}_\sharp}(w_1 - w_2)) = N(f_\sharp(w_1 - w_2, w_1 - w_2)) = N(-1) = -1 = \boxtimes$. Thus $D(x_i{}^{\perp}_{V}) = \boxtimes$ and $D(y_i{}^{\perp}_{V}) = \square$ for all $i = 1, \ldots, 3$, and so by [21, Proposition 2.5.11], $sgn(x_i{}^{\perp}_{V}) = (-)^{m-1} = (-)^b$ and $sgn(y_i{}^{\perp}_{V}) = (-)^m = (-)^{3b+1} = (-)^{b-1}$. Let $z \in \{x_i, y_i\}$ and $\gamma = Q_\sharp(z)$. We have $\langle z\rangle M_\Omega = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) \in \{\gamma, \gamma^3, \gamma^9\}\} = \bigcup_{j=1}^{3}\langle z\phi^j_\sharp\rangle\Omega_\sharp$. Observe that in $(V_\sharp, \mathbb{F}_\sharp, Q_\sharp)$ all vectors $z\phi^j_\sharp, j = 1, \ldots, 3$ have the same type, say $\varepsilon = sgn(z^{\perp}_{V_\sharp})$. It follows from Lemma 2.2 that $1 + c_z + d_z = 3\frac{1}{2}(q^{2b} + \varepsilon q^b) = \frac{1}{2}(3^{6b+1} + \varepsilon 3^{3b+1})$. For any $w \in \langle z\rangle M_\Omega \cap z^{\perp}_{V}$, $Q_\sharp(w) \in \{\gamma, \gamma^3, \gamma^9\}$ and $T(\varphi) = 0$, with $\varphi = f_\sharp(w,z)$. Write $w = \varphi f_\sharp(z,z)^{-1}z + w_0$, where $w_0 \in z^{\perp}_{V_\sharp}$. We have $f_\sharp(w,z) = \varphi$ and $Q_\sharp(w_0) = \gamma^{-1}(\gamma Q_\sharp(w) - \varphi^2)$.

Assume that $i = 1, 2$. Let $z \in \{x_i, y_i\}$ and $\gamma = Q_\sharp(z)$. Then $sgn(z_V^\perp) = sgn(z_{V_\sharp}^\perp) = \varepsilon$. We will show that $Q_\sharp(w_0) \neq 0$ for any $\varphi \in \ker T$. By way of contradiction, suppose that $Q_\sharp(w_0) = 0$. Then $\varphi^2 \in \{\gamma^2, \gamma^4, \gamma^{10}\}$, hence $\varphi \in \{\pm\gamma, \pm\gamma^2, \pm\gamma^5\}$ or $\varphi \in \{\pm\omega^2, \pm\omega^4, \pm\omega^8, \pm\omega^{20}, \pm\omega^{10}\}$. As the trace map is non-zero on these values, we get a contradiction. Thus $Q_\sharp(w_0) \neq 0$. By Lemma 2.2, $d_z = 3\frac{1}{2}(q^{2b-1} - \varepsilon q^{b-1}) \cdot 3^{\alpha-1} = \frac{1}{2}(q^{2b} - \varepsilon q^b)$ as $|\ker T| = 3^{\alpha-1} = 3^2$, $\dim_{\mathbb{F}_\sharp}(z_{V_\sharp}^\perp) = 2b$, and $\varepsilon = sgn(z_{V_\sharp}^\perp)$. Then $c_z = q^{2b} + \varepsilon 2q^b - 1$. Hence $c_z - 2d_z = \varepsilon 3q^b - 1 = \varepsilon 3^m - 1$. Therefore equation (5.1) holds.

Assume $z \in \{x_3, y_3\}$. Then $sgn(z_V^\perp) = -sgn(z_{V_\sharp}^\perp) = -\varepsilon$, $\gamma = \pm\omega^8$ and $Q_\sharp(w_0) = \gamma^{-1}(\gamma Q_\sharp(w) - \varphi^2)$. For any $w \in \langle z \rangle M_\Omega$, $Q_\sharp(w) \in \{\gamma, \gamma^3, \gamma^9\}$. If $Q_\sharp(w) = \gamma$ then $Q_\sharp(w_0) \neq 0$ as $T(\varphi) = T(\pm\gamma) \neq 0$. If $Q_\sharp(w) = \gamma^3$ then $Q_\sharp(w_0) = 0$ if and only if $\varphi \in \{\pm\gamma^2\}$, and similarly, if $Q_\sharp(w) = \gamma^9$ then $Q_\sharp(w_0) = 0$ if and only if $\varphi \in \{\pm\gamma^5\}$. By Lemma 2.2, we have $2d_z = 3^2 \cdot (q^{2b-1} - \varepsilon q^{b-1}) + 2[2(q^{2b-1} + \varepsilon(q^b - q^{b-1})) + 7(q^{2b-1} - \varepsilon q^{b-1})]$. After simplifying, we get $2d_z = 3^{6b} + \varepsilon 3^{3b+1}$, and hence $c_z = 3^{6b} - 1$, so that $c_z - 2d_z = -\varepsilon 3^{3b+1} - 1 = -\varepsilon 3^m - 1$. Thus equation (5.1) holds.

Let $\xi = (-)^b = sgn(x_{iV}^\perp)$ and $\eta = sgn(y_{iV}^\perp), i = 1, \ldots, 3$. As $\sum_{i=1}^3 |\langle x_i \rangle M_\Omega| = \frac{1}{2}(3^{6b+1} + \xi 3^{3b+1}) + \frac{1}{2}(3^{6b+1} + \xi 3^{3b+1}) + \frac{1}{2}(3^{6b+1} - \xi 3^{3b+1}) = \frac{1}{2}3^{3b+1}(3^{3b+1} + \xi 1) = \frac{1}{2}3^m(3^m + \xi 1) = |\mathfrak{E}_\xi(V)|$, M_Ω has exactly three orbits on $\mathfrak{E}_\xi(V)$. Thus equation (5.1) holds for all points in $\mathfrak{E}_\xi(V)$. Similarly M_Ω has three orbits on $\mathfrak{E}_\eta(V)$, and so equation (5.1) holds for all points in $\mathfrak{E}_\eta(V)$. □

5.2.4. *The tensor product subgroups $\mathcal{C}_4$.* Let V_i be vector spaces over $\mathbb{F}$ of dimension n_i, $i = 1, \ldots, t$. Let $V = V_1 \otimes \cdots \otimes V_t$. For $g_i \in GL(V_i)$, the element $g_1 \otimes \cdots \otimes g_t \in GL(V)$ acts on V as follows: $(v_1 \otimes \cdots \otimes v_t)(g_1 \otimes \cdots \otimes g_t) = v_1 g_1 \otimes \cdots \otimes v_t g_t (v_i \in V_i)$, and extends linearly. Now suppose that $\mathbf{f}_i$ is a non-degenerate symmetric bilinear form on V_i, so that $(V_i, \mathbb{F}, \mathbf{f}_i)$ is an orthogonal geometry. We next define the bilinear form $\mathbf{f} = \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_t$ on $V_1 \otimes \cdots \otimes V_t$ by $\mathbf{f}(v_1 \otimes \cdots \otimes v_t, w_1 \otimes \cdots \otimes w_t) = \prod_{i=1}^t \mathbf{f}_i(v_i, w_i)$ and extend linearly. We write $(V, \mathbf{f}) = (V_1 \otimes \cdots \otimes V_t, \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_t)$ for such a structure and call a *tensor decomposition* and denote by $\mathcal{D}$. The members of $\mathcal{C}_4(\Xi)$ is the stabilizer of a tensor decomposition $\mathcal{D}$ such that

(*a*) $(V, \mathbf{f}) = (V_1 \otimes V_2, \mathbf{f}_1 \otimes \mathbf{f}_2)$,

(*b*) $(V_1, \mathbf{f}_1)$ is not similar to $(V_2, \mathbf{f}_2)$,

(*c*) $\mathbf{f}_i$ are symmetric, $(n_1, \varepsilon_1) \neq (n_2, \varepsilon_2)$, where $n_i = dimV_i \geqslant 3$, $\varepsilon_i = sgnV_i$, and $dimV_1 < dimV_2$.

Proposition 5.14. *Assume M is of type $O_{n_1}(3) \otimes O_{n_2}(3)$, with $n_1 < n_2$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1-3.*

Proof. Let $v = v_1 \otimes v_2 \in V$, where $v_i \in V_i$ are non-singular vectors. Then $\langle v \rangle$ is a non-singular point. By [21, (4.4.14)], $M_I = I_1 \otimes I_2$. We have $\Omega_1 \times \Omega_2 \trianglelefteq M_\Omega \leqslant I_1 \otimes I_2 = M_I$. As all Ω_i act transitively on $\mathfrak{E}_{\rho_{V_i}(v_i)}(V_i)$, it follows that $\langle v \rangle M_I = \langle v \rangle M_\Omega = \langle v_1 \otimes v_2 \rangle(\Omega_1 \times \Omega_2) = \langle v_1 \rangle \Omega_1 \otimes \langle v_2 \rangle \Omega_2$. Therefore $\langle v \rangle M_\Omega = \langle v \rangle M$ since $M_\Omega \leqslant M \leqslant M_I$. Write $n_1 = 2a + 1, n_2 = 2b + 1$. Then $|\langle v \rangle M_\Omega| \leqslant |\langle v_1 \rangle \Omega_1||\langle v_2 \rangle \Omega_2| \leqslant \frac{1}{2}3^a(3^a + 1)3^b(3^b + 1) < 3^{2(a+b)}$ as $3^a + 3^b + 1 < 3^{a+b}$ and $b > a \geqslant 1$. Since $dimV = 2m + 1 = (2a + 1)(2b + 1)$, $m = 2ab + a + b$. Then $m - 1 - 2(a + b) = a(b - 2) + b(a - 1) + a - 1 \geqslant 0$ so that $3^{m-1} \geqslant 3^{2(a+b)} > |\langle v \rangle M_\Omega|$. This violates (5.3) so that (3.1) cannot hold. □

5.2.5. *The tensor product subgroups* $\mathcal{C}_7$. Let V_1 be an α-dimensional vector space over $\mathbb{F}$, and assume that $\mathbf{f}_1$ is a non-degenerate symmetric bilinear form on V_1. For $i = 1, \ldots, b$, let $(V_i, \mathbf{f}_i)$ be a classical geometry which is similar to $(V_1, \mathbf{f}_1)$. For each i, denote by η_i the similarity from $(V_1, \mathbf{f}_1)$ to $(V_i, \mathbf{f}_i)$ satisfying $\mathbf{f}_i(v\eta_i, w\eta_i) = \lambda_i \mathbf{f}_1(v, w)$ for all $v, w \in V_1$, where $\lambda_i \in \mathbb{F}^*$ is independent of v and w. Thus we obtain a tensor decomposition $\mathcal{D}$ given by $(V, \kappa) = (V_1, \mathbf{f}_1) \otimes \cdots \otimes (V_b, \mathbf{f}_b)$, where $V = V_1 \otimes \cdots \otimes V_b$ and $\kappa = \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_b$. Let $X_i = X(V_i, \mathbf{f}_i)$ for $X \in \{\Omega, S, I, \Lambda, \Xi, A\}$. Define $\Xi_{\mathcal{D}} = \Xi_{(\mathcal{D})} S_b$. The members of $\mathcal{C}_7(\Xi)$ are the groups $\Xi_{\mathcal{D}}$ with $b \geqslant 2$ described as above.

Proposition 5.15. *Assume M is of type $O_\alpha(3) \wr S_b$ with $\alpha \geqslant 5$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that (3.1) does not hold so that M is not in Tables 1-3.*

Proof. We have $\Omega_\alpha(3) \wr S_b \trianglelefteq M_\Omega \leqslant O_\alpha(3) \wr S_b$. Let $v_1 = v \otimes v \otimes \cdots \otimes v$ and $v_2 = v_1 + w \otimes w \otimes \cdots \otimes w \in V$, where $v \neq w$ belong to some orthogonal basis of V_1. As S_b fixes v_i, $\langle v_i \rangle M_\Omega = \langle v_i \rangle (\Pi_{i=1}^b \Omega_\alpha(3))$. Also $\langle v_i \rangle M_\Omega = \langle v_i \rangle M_I = \langle v_i \rangle M$. For $i = 1, 2$, the stabilizers of $\langle v_i \rangle$ in $\Pi_{i=1}^b \Omega_\alpha(3)$ contain a subgroup which is isomorphic to $\Pi_{i=1}^b \Omega_{\alpha-2}(3)$. Thus $|\langle v_i \rangle M_\Omega| = |\langle v_i \rangle (\Pi_{i=1}^b \Omega_\alpha(3))| \leqslant [\Omega_\alpha(3) : \Omega_{\alpha-2}(3)]^b < 3^{(4a-3)b}$, where $a = \frac{\alpha-1}{2} \geqslant 2$. Now $m - 1 = \frac{1}{2}((2a+1)^b - 3)$. Consider the following function in variable $x \in [2, +\infty)$, where $b \geqslant 2$, $f(x) := \frac{1}{2}((2x+1)^b - 3) - (4x-3)b$. We have $f'(x) = b(2x+1)^{b-1} - 4b \geqslant b(2x+1) - 4b \geqslant b > 0$. Hence $f(x) \geqslant f(2) = \frac{1}{2} g(b)$, where $g(b) = 5^b - 10b - 3$ and $b \geqslant 2$. By induction on $b \geqslant 2$, $g(b) > 0$. Thus $3^{m-1} > 3^{b(4a-3)} > |\langle v_i \rangle M_\Omega|$. This contradicts (5.3). Thus (3.1) cannot hold. □

These are all the maximal subgroups in $\mathcal{C}(G)$ of G. We next consider maximal subgroups in $\mathcal{S}(G)$.

5.3. **Permutation characters of maximal subgroups in $\mathcal{S}$.** In this section, we consider the maximal subgroup $M \in \mathcal{S}(G)$. By definition of $\mathcal{S}$, M is an almost simple group and the socle S of M is a non-abelian simple group. Then the full covering group $\widehat{S}$ of S acts absolutely irreducible on V, the natural module for G and preserves a non-degenerate quadratic form on V, that is, $ind(V) = +$.

We note that some of the small cases in this section will be handled by using GAP [10]. We will describe how to do this at the end of the paper.

5.3.1. *Embedding of alternating and symmetric groups.* Recall the construction of the fully deleted permutation module for alternating groups in [21, p. 185]. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be a standard basis for $\mathbb{F}_p^n$, and let $w_0 = \varepsilon_1 + \cdots + \varepsilon_n \in \mathbb{F}_p^n$. Put $U = w_0^\perp, W = \mathbb{F}_p w_0$, and $V = U/(U \cap W)$. Define $e_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1$. Then $\{e_i\}_{i=1}^{n-1}$ is a basis for V if p does not divide n, and $\{e_i + U \cap W\}_{i=1}^{n-2}$ is a basis for V if $p|n$. Define $\varepsilon_p(n)$ to be 1 if n is divisible by p, otherwise, $\varepsilon_p(n) = 0$. Then $dimV = n - 1 - \varepsilon_p(n)$. Let Q be the quadratic form on V induced from the quadratic form associated to the natural bilinear form on $\mathbb{F}_p^n$. Then $(V, \mathbb{F}_p, Q)$ is a classical orthogonal geometry and $A_n \leqslant \Omega(V)$. To simplify the notation, we always write e_i instead of $e_i + U \cap W$.

Lemma 5.16. *Assume M is almost simple of type A_n with $n \geqslant 10$ and V is the fully deleted permutation module for A_n in characteristic* 3. *Let $v = \varepsilon_1 - \varepsilon_2, w = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \in V$. Then*

(1) $|\langle v \rangle M| = \frac{1}{2} n(n-1), d_v = \frac{1}{2}(n-2)(n-3),$ *and* $c_v = 2n - 4$;

(2) $|\langle w \rangle M| = \frac{1}{8} n(n-1)(n-2)(n-3), c_w = 2n^3 - 25n^2 + 111n - 172$ *and* $d_w = 2 + 4(n-4)^2 + \frac{1}{8}(n-4)(n-5)(n-6)(n-7)$.

Proof. Observe that $\langle v\rangle S_n = \{\langle \varepsilon_i - \varepsilon_j\rangle \mid i \neq j \in \{1,\dots,n\}\}$ and $\langle w\rangle S_n$ is the set of all points of the form $\langle \varepsilon_i + \varepsilon_j - \varepsilon_r - \varepsilon_s\rangle$, where $i,j,r,s \in \{1,\dots,n\}$ and pair-wise distinct. As $n \geqslant 10$ and A_n is $(n-2)$-transitive on the index set $\{1,\dots,n\}$, we have $\langle x\rangle A_n = \langle x\rangle S_n$ for any $x \in \{v,w\}$. Hence $\langle x\rangle M = \langle x\rangle S_n$ for $x \in \{v,w\}$. Since $v = \varepsilon_1 - \varepsilon_2$, it is clear that if $g \in S_n$ and $(\varepsilon_1-\varepsilon_2)g = \varepsilon_1 - \varepsilon_2$, then g must fix indices 1 and 2. Thus $(S_n)_v \simeq S_{n-2}$. Similarly, $(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)g = \varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4$ implies that g must fix the partitions $\{1,2\},\{3,4\}$. Thus $g \in S_2 \times S_2 \times S_{n-4}$. Therefore $|M : M_{\langle v\rangle}| = \frac{1}{2}[S_n : S_{n-2}] = \frac{1}{2}n(n-1)$, and $|M : M_{\langle w\rangle}| = \frac{1}{2}[S_n : S_2 \times S_2 \times S_{n-4}] = \frac{1}{8}n(n-1)(n-2)(n-3)$.

(i) Parameters for v. We have $\langle u\rangle \in \langle v\rangle M \cap v^{\perp}$ if and only if $u = \varepsilon_i - \varepsilon_j \notin \mathbb{F}_3 v, i \neq j$ and $(\varepsilon_i - \varepsilon_j, \varepsilon_1 - \varepsilon_2) = 0$. This happens only if $\{i,j\} \cap \{1,2\} = \emptyset$, or $i,j \in \{3,4,\dots,n\}$. There are $\binom{n-2}{2}$ such points $\langle u\rangle$. Thus $d_v = |\langle v\rangle M \cap v^{\perp}| = \frac{1}{2}(n-2)(n-3)$, and $c_v = 2n-4$.

(ii) Parameters for w.
We will show that $c = 2n^3 - 25n^2 + 111n - 172$ and $d = |\langle w\rangle M \cap w^{\perp}| = 2 + 4(n-4)^2 + \frac{1}{8}(n-4)(n-5)(n-6)(n-7)$. For any $\langle u\rangle \in \langle w\rangle M \cap w^{\perp}$, $u = \varepsilon_i + \varepsilon_j - \varepsilon_r - \varepsilon_s$, where $|\{i,j,r,s\}| = 4$, and $(u,w) = 0$. Denote by $supp(u)$ the set of non-zero indices of ε_i appearing in u. We consider the cases:

(1) $supp(u) \cap supp(w) = \emptyset$. Then $supp(u) \in \{5,6,\dots,n\}$. Hence there are $(n-4)(n-5)(n-6)(n-7)/8$ points.

(2) $|supp(u) \cap supp(w)| = 1$. There are no such u, since $(u,w) \neq 0$.

(3) $|supp(u) \cap supp(w)| = 2$. Suppose that $i,j \in \{1,2,3,4\}$. Then either $u = \varepsilon_i - \varepsilon_j + \varepsilon_r - \varepsilon_s$, where $\varepsilon_i - \varepsilon_j \in \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4\}$, or $u = \varepsilon_i + \varepsilon_j - \varepsilon_r - \varepsilon_s$, where $\varepsilon_i + \varepsilon_j \in \{\varepsilon_1+\varepsilon_3, \varepsilon_1+\varepsilon_4, \varepsilon_2+\varepsilon_3, \varepsilon_2+\varepsilon_4\}$; and $r,s \in \{5,\dots,n\}$. There are $2(n-4)(n-5)$ and $4\binom{n-4}{2}$ points respectively. Thus there are $4(n-4)(n-5)$ points in this case.

(4) $|supp(u) \cap supp(w)| = 3$. Suppose that $i,j,r \in \{1,2,3,4\}$. Then $u = \pm\varepsilon_i \pm \varepsilon_j \pm \varepsilon_r \pm \varepsilon_s$, where $s \in \{5,\dots,n\}$, $\varepsilon_i, \varepsilon_j, \varepsilon_r$ with their signs appearing exactly as in w, and sign of ε_s is chosen so that there are 2 minuses and 2 pluses. There are $\binom{4}{3}(n-4) = 4(n-4)$ such points.

(5) $|supp(u) \cap supp(w)| = 4$. There are just 2 points in this case: $\{\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4, \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4\}$.

Therefore $d_w = \frac{1}{8}(n-4)(n-5)(n-6)(n-7) + 2(n-4)(n-5) + 4(n-4)(n-5) + 2 = \frac{1}{8}(n-4)(n-5)(n-6)(n-7) + 2(n-4)^2 + 2$, and $c_w = \frac{1}{8}n(n-1)(n-2)(n-3) - d - 1$. □

Proposition 5.17. *Assume M is almost simple of type A_n, with $n \geqslant 10$, and V is the fully deleted permutation module for A_n in characteristic $p = 3$. Further assume that $n - 1 - \varepsilon_3(n) = 2m+1$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1-3.*

Proof. Let $v = \varepsilon_1 - \varepsilon_2, w = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \in V$. Then $Q(v) = 1$, and $Q(w) = -1$. Hence v, w are non-singular vectors in V. We see that $n - 1 - \varepsilon_3(n)$ is odd if and only if $n = 6k+2, n = 6k+3$ or $n = 6k+4$. By Lemma 5.16(1), $|\langle v\rangle M| = \frac{1}{2}n(n-1)$.

Assume that $n \geqslant 13$. Then $m - 1 = \frac{1}{2}(n - 2 - \varepsilon_3(n)) - 1 \geqslant \frac{1}{2}(n-5)$, as $\varepsilon_3(n) \leqslant 1$, and so $3^{m-1} \geqslant 3^{(n-5)/2} > n(n-1)/2 = 1 + c + d$, violating (5.3) and so equation (3.1) cannot hold.

Hence we can assume that $10 \leqslant n \leqslant 12$. Since $n = 6k+2, 6k+3$ or $6k+4$, it follows that $n = 10$. Then $n - 1 - \varepsilon_3(n) = 9, d_v = 28, c_v = 16, m = 4$. If equation (3.1) holds, then either $c_v - 2d_v = \xi 3^4 - 1 = \xi 81 - 1 = -40$, or $(\xi 27 + 1)(\xi 81 - 41) = 32$.

TABLE 5. Small degree representations of ssome alternating groups.

A_n	A_6	A_7	A_7	A_8	A_8	A_8	A_9	A_9
$\dim D^\lambda$	9	13	15	7	13	21	21	7
λ	$(4,2)$	$(5,2)$	$(5,1^2)$	$(7,1)$	$(6,2)$	$(6,1^2)$	$(7,1^2)$	$(8,1)$
$m(\lambda)$	$(2^2,1^2)$	$(3,2,1^2)$	$(3,2^2)$	$(4,3,1)$	$(3^2,1^2)$	$(3^2,2)$	$(4,3,2)$	$(4^2,1)$

These equations clearly cannot hold with $\xi = \pm$. By Lemma 5.16(2), $|\langle w\rangle M| = n(n-1)(n-2)(n-3)/8$. If $n \geqslant 23$, then $(3^m+1)/2 > n(n-1)(n-2)(n-3)/8$ so that equation (3.1) cannot hold by (5.3). Thus we can assume that $10 \leqslant n \leqslant 22$. Then $n \in \{10, 14, 15, 16, 20, 21, 22\}$.
(a) $n = 10$. Then $n - \varepsilon_3(n) = 9, m = 4,\ d = 191, c = 438$ and $c - 2d = 56$.
(b) $n = 14$. Then $n - \varepsilon_3(n) = 13, m = 6,\ d = 1032, c = 1970$ and $c - 2d = -94$.
(c) $n = 15$. Then $n - \varepsilon_3(n) = 13, m = 6,\ d = 1476, c = 2618$ and $c - 2d = -334$.
(d) $n = 16$. Then $n - \varepsilon_3(n) = 15, m = 7,\ d = 2063, c = 3396$ and $c - 2d = -730$.
(e) $n = 20$. Then $n - \varepsilon_3(n) = 19, m = 9,\ d = 6486, c = 8048$ and $c - 2d = -4924$.
(f) $n = 21$. Then $n - \varepsilon_3(n) = 19, m = 9,\ d = 8298, c = 9656$ and $c - 2d = -6940$.
(g) $n = 22$. Then $n - \varepsilon_3(n) = 21, m = 10,\ d = 10478, c = 11466$ and $c - 2d = -9490$.
We can check that equation (3.1) cannot hold in any of these cases. □

Proposition 5.18. *Assume M is almost simple of type A_n with $n \geqslant 12$, and V is not the fully deleted permutation module for A_n in characteristic $p = 3$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1-3.*

Proof. As $n \geqslant 12$, by Lemma 2.6, we have $dim(V) = 2m+1 \geqslant (n^2-5n+2)/2$ so that $m \geqslant (n^2-5n)/2$. However when $n \geqslant 12$, $3^{m-1} \geqslant 3^{(n^2-5n-4)/4} > n! = |Aut(A_n)|$. Thus equation (3.1) cannot hold in view of (5.3). □

Let $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_h^{a_h})$ be a p-regular partition. Then λ is called a JS-partition if $\lambda_i - \lambda_{i+1} + a_i + a_{i+1} \equiv 0 \pmod p$.

Proposition 5.19. *Assume M is almost simple of type $S = A_n$ with $5 \leqslant n \leqslant 11$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $n = 9$ and $V \cong D^{(8,1)}$, in which case M has at most 2 orbits on $\mathfrak{E}_\xi(V)$ so that $1_P^G \nleqslant 1_M^G$ by Corollary 3.7 and hence $(L, S) = (\Omega_7(3), A_9)$ is in Table 2.*

Proof. Using information on the p-modular representations of alternating groups and their covering groups in [20], we need to consider the cases given in Table 5. (i) Let $\lambda = (8,1)$. Then $m(\lambda) = (4^2,1) \neq \lambda$. By [7, Theorem 2.1], $D^\lambda \downarrow_{A_9}$ is irreducible. Thus $A_9 \leqslant S_9 \leqslant \Omega_7(3)$, and there are two classes of S_9 in $\Omega_7(3)$. As $8 - 1 + 1 + 1 = 9 \equiv 0 (\mathrm{mod}\ 3)$, λ is a JS-partition, and hence by [23, Theorem 0.3], $D^\lambda \downarrow_{S_8} = D^{\lambda(1)} = D^{(7,1)}$. Then since $(7,1) \neq m(7,1) = (4,3,1)$, we have: $A_8 \leqslant S_8 \leqslant S_9$. In this case, D^λ is the fully deleted permutation module for S_9 over $\mathbb{F}_3$. Then $n - 1 - \varepsilon_3(n) = 7$, and $m = 3$. Let $v = \varepsilon_1 - \varepsilon_2, w = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \in V$. There are only two orbits of type $\rho_V(v)$, with representatives $v = \varepsilon_1 - \varepsilon_2$ and $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8$, and one orbit of type $\rho_V(w)$. Thus equation (3.1) holds for both types of points.

(ii) If $\lambda = (7,1)$ or $\lambda = (4,3,1)$, then $A_8 < S_8 < S_9 < \Omega_7(3)$, since $D^{(8,1)} \downarrow_{S_8} = D^{(7,1)}$ and $D^{(7,1)} \downarrow_{A_8}$ is irreducible.

(iii) If $\lambda = (6, 1^2)$ or $(3^2, 2)$, then $A_8 < S_8 < S_9 < \Omega_{21}(3)$, since $D^{(7,1^2)} \downarrow_{S_8} = D^{(6,1^2)}$ and $D^{(6,1^2)} \downarrow_{A_8}$ is irreducible.

(iv) If $\lambda = (5, 2)$ or $(3, 2, 1^2)$, then $A_7 < S_7 < S_8 < \Omega_{13}(3)$.

(v) Now, if $\lambda = (n-2, 1^2)$, where $n = 7$ or 9, then $D^\lambda = \wedge^2(D^{(n-1,1)})$. As $D^{(n-1,1)}$ is the fully deleted permutation module for S_n, we can apply the construction above for fully deleted module. Let $v = e_1 \wedge e_3 = (\varepsilon_1 - \varepsilon_2) \wedge (\varepsilon_3 - \varepsilon_4)$, and $w = (e_1 - e_2) \wedge (e_3 - e_4) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \wedge (\varepsilon_3 + \varepsilon_4 + \varepsilon_5)$. Then v, w are non-singular points of different types in D^λ. We have $|vS_n| = \frac{1}{2}\frac{n!}{2(n-4)!}$ and $|wS_n| = \frac{1}{2}\frac{n!}{2(n-5)!}$. We then get contradictions by using (5.3).

(vi) If $\lambda = (4, 2)$ then $dimD^\lambda = 9, m = 4$, and we have an embedding $A_6 \leqslant S_6 \leqslant \Omega_9(3)$. As $A_6 \cong L_2(9) < A_{10} < \Omega_9(3)$, $M = N_G(A_6)$ is not maximal in G.

(vii) If $\lambda = (6, 2)$, then $dimV = 13, m = 6$. We have $A_8 \leqslant S_8 \leqslant \Omega_{13}(3)$. Using [10], S_8 has two points w_1, w_2 of different types with the same orbit sizes 315. We have $(c, d) = (230, 84)$ or $(c, d) = (212, 102)$. We see that equations (5.1) and (5.2) cannot hold. □

5.3.2. *Embedding of groups of Lie types in cross-characteristic.*

Proposition 5.20. *Assume M is almost simple of type S, where S is a finite simple group of Lie type in cross-characteristic. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $(L, S) = (\Omega_7(3), PSp_6(2))$, in which case M has only two orbits on $\mathfrak{E}_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7, and so (L, S) is in Table 2.*

Proof. Suppose equation (3.1) holds for some $r \in \{s, t\}$ and some M-orbit $\langle x\rangle M$ with $\langle x\rangle \in \mathfrak{E}_\xi(V)$. Then $|\langle x\rangle M| \geqslant \frac{1}{2}(3^m + 1)$, by (5.3). On the other hand, using the lower bounds for degrees of cross-characteristic representations of finite simple groups of Lie type (see [24] and [31]), we have $2m+1 \geqslant e(S)$, so that $(3^m+1)/2 \geqslant (3^{(e(S)-1)/2})/2$. Moreover $|\langle x\rangle M| \leqslant |M| \leqslant |Aut(S)|$. It follows that $(3^{\frac{e(S)-1}{2}}+1)/2 \leqslant |Aut(S)|$. By this condition, we get a finite list as in Table 6. Next we shorten the list by using information on cross-characteristic representations of small groups together with [20] and [10]. Further V is an absolutely irreducible $\mathbb{F}_3\widehat{S}$-module which is self-dual with $ind(V) = +1$, and $dimV$ is odd.

(i) Case $S = L_2(q), 2 \leqslant q \leqslant 68$. As $L_2(2), L_2(3)$ are not simple, $L_2(4) \cong L_2(5) \cong A_5$, and q is a prime power, we can assume that $7 \leqslant q \leqslant 67$.

Case $q \equiv 1 \pmod 4$. By [14, Table 2] and the fact that $dimV$ is odd, we have $dimV \in \{(q+1)/2, q\}$. Using (5.3) again, we only need to consider the following cases:

(1) $q \in \{13, 17, 29, 37, 41, 49\}$ and $dimV = (q+1)/2$;

(2) $q = 13, 17$ and $dimV = q$.

If $dimV = (q+1)/2$, then q must be a square in $\mathbb{F}_3$, so that $q \equiv 1 \pmod 3$ and hence $q = 13, 37, 49$. If $dimV = q$, then $3 \nmid (q+1)$, so that $q = 13$.

If $(S, dimV) = (L_2(13), 7)$ then $L_2(13) < G_2(3) < \Omega_7(3)$ by [5] so M is not maximal in G.

If $(S, dimV) = (L_2(13), 13)$ then $L_2(13) < \Omega_{13}(3)$. Let w be a non-singular eigenvector of an element of order 13. Then $|\langle w\rangle S| = 14$ and hence $|\langle w\rangle M| \leqslant |Out(S)||\langle w\rangle S| = 2 \cdot 14 < 3^{m-1} = 3^5$ so that equation (3.1) cannot hold. For other type of point, if $M = S$ then there exists a point $\langle u\rangle$ with $|\langle u\rangle S| = 1092$ and $(c, d) = (734, 357)$. If $M = S \cdot 2$ then there exists a point $\langle v\rangle$ with $|\langle v\rangle M| = 2184$

TABLE 6. Small groups in cross-characteristic.

S	$(3^{\frac{e(S)-1}{2}}+1)/2 \leqslant \lvert Aut(S)\rvert$
$L_2(q)$	$L_2(q), 3 \leqslant q \leqslant 68$
$L_n(q), n \geqslant 3$	$L_3(2), L_4(2), L_5(2), L_3(4)$
$PSp_{2n}(q), n \geqslant 2$	$S_4(5), S_4(7), S_6(5)$
	$S_4(2), S_6(2), S_8(2), S_4(4)$
$U_n(q), n \geqslant 3$	$U_n(2), 3 \leqslant n \leqslant 7, U_3(4), U_3(5)$
$P\Omega^+_{2n}(q), n \geqslant 4$	$\Omega^+_8(2)$
$P\Omega^-_{2n}(q), n \geqslant 4$	$\Omega^-_8(2)$
$\Omega_{2n+1}(q), n \geqslant 3, q$ odd	
$E_6(q)$	
$E_7(q)$	
$E_8(q)$	
$F_4(q)$	$F_4(2)$
${}^2E_6(q)$	
$G_2(q)$	
${}^3D_4(q)$	${}^3D_4(2)$
${}^2F_4(q)$	${}^2F_4(2)'$
$Sz(q)$	$Sz(8)$
${}^2G_2(q)$	

and $(c, d) = (1469, 714)$. We check that equation (3.1) cannot hold in any of these cases.

If $(S, dimV) = (L_2(37), 19)$ then $m = 9$ and the eigenvector w of an element of order 37 is non-singular and $|\langle w\rangle S| = 38$, $|Out(S)| = 2$ and hence $|\langle w\rangle M| \leqslant 38 \cdot 2 < 3^{m-1}$, so that equation (3.1) cannot hold for this point. For other type of point, there exists a point $\langle u\rangle$ with $|\langle u\rangle S| = |S| = 25308$ and $(c, d) = (16919, 8388)$. We see that equation (3.1) cannot hold.

If $(S, dimV) = (L_2(49), 25)$ then $m = 12$ and there exist two non-singular vectors of different type $u_+, u_- \in V$ which are eigenvectors of an element of order 7 in S such that $|\langle u_\xi\rangle S| \leqslant 8400, \xi = \pm$. As $|Out(S)| = 4$, the latter inequality yields $|\langle u_\xi\rangle M| \leqslant |Out(S)||\langle u_\xi\rangle S| \leqslant 4 \cdot 8400 < 3^{m-1}$. In view of (5.3), equation (3.1) cannot hold.

Case $q \equiv 3 \pmod 4$. As in previous case, by [14, Table 2], we have $dimV = q$ and $3 \nmid q + 1$. Using (5.3) again, we get $q \in \{7, 19\}$.

If $(S, dimV) = (L_2(7), 7)$ then $L_2(7) < \Omega_7(3)$ but $\Omega_7(3)$ has no maximal subgroup with socle $L_2(7)$ by [5].

If $(S, dimV) = (L_2(19), 19)$ then $m = 9$ and there exist two non-singular vectors of different type $u_+, u_- \in V$ which are eigenvectors of an element of order 5 in S such that $|\langle u_\xi\rangle S| \leqslant 342, \xi = \pm$. As $|Out(S)| = 2$, this implies $|\langle u_\xi\rangle M| \leqslant |Out(S)||\langle u_\xi\rangle S| \leqslant 2 \cdot 342 < 3^{m-1}$. In view of (5.3) equation (3.1) cannot hold.

Case $q \equiv 0 \pmod 2$. As in previous case, by [14, Table 2], we have $dimV \in \{q-1, q+1\}$. Using (5.3) again, we have $q = 8, 16$. By [20], the only possibility is $q = 8$ and $dimV = 7$. However by [5], $L_2(8) \leqslant G_2(3) \leqslant \Omega_7(3)$.

(ii) Case $L_n(q), (n, q) \in \{(3, 2), (4, 2), (5, 2), (3, 4)\}$. As $L_3(2) \cong L_2(7)$, $L_4(2) \cong A_8$, which have been done above, we can exclude these groups.

If $S = L_3(4)$, then $Out(S) \cong 2 \times S_3 \cong D_{12}$. By [20], $dimV \in \{15, 19, 45, 63\}$. By using (5.3), we only need to consider the representations of degrees 15 and 19.

Assume first that $dimV = 15$. Then $m = 7$. If $M = L_3(4)$ then there exists two non-singular vectors $u_i, i = 1, 2$ with $|\langle u_i \rangle L_3(4)| = 2016$, $(c_1, d_1) = (1250, 765)$, and $(c_2, d_2) = (1350, 660)$. Similarly if $M = L_3(4) \cdot 2_1$ then $(c_1, d_1) = (2600, 1431)$, $(c_2, d_2) = (1250, 765)$; if $M = L_3(4) \cdot 2_2$ or $L_3(4).2_3$ then $(c_1, d_1) = (2720, 1311)$, $(c_2, d_2) = (2810, 1221)$; if $M = L_3(4) \cdot 2^2$ then $(c_1, d_1) = (1250, 765)$, $(c_2, d_2) = (5327, 2736)$. We check that equation (3.1) cannot hold.

Assume that $dimV = 19$. Then $m = 9$. If $M = L_3(4)$ then there exist two non-singular vectors of different type $u_i, i = 1, 2$ which are the eigenvectors of elements of order $7, 5$, respectively and $(c_1, d_1) = (707, 252), (c_2, d_2) = (1190, 825)$. For the remaining extensions of $L_3(4)$, there exist two non-singular vectors of different type $u_i, i = 1, 2$, which are the eigenvectors of an element of order 5 such that the parameters $(c_i, d_i), i = 1, 2$, are as follows: If $M = L_3(4) \cdot 2_2$ or $M = L_3(4) \cdot 2_3$ then $(c_1, d_1) = (1190, 825), (c_2, d_2) = (1100, 915)$. If $M = L_3(4) \cdot 2_3$ then $(c_1, d_1) = (2561, 1470), (c_2, d_2) = (1100, 915)$. If $M = L_3(4) \cdot S_3$ or $L_3(4) \cdot D_{12}$ then $(c_1, d_1) = (3842, 2205), (c_2, d_2) = (4220, 1827)$. If $M = L_3(4) \cdot 3$ then $(c_1, d_1) = (3932, 2115), (c_2, d_2) = (4220, 1827)$. We can check that equation (3.1) cannot hold in any of these cases.

Finally if $S = L_5(2)$, then $dimV \geqslant 155$. But $(3^m + 1)/2 \geqslant (3^{77} + 1)/2 > |Aut(L_5(2))|$ so that equation (3.1) cannot hold.

(iii) Case $S \in \{S_4(5), S_4(7), S_6(5)\}$. If $S \cong S_4(5)$ or $S_6(5)$ then the smallest odd degree non-trivial irreducible representations of S has degree 13, and 63, respectively. However since the smallest field of definitions of these representations are quadratic extensions of $\mathbb{F}_3$, (cf. [14]), L cannot embed in $\Omega_{13}(3)$ and $\Omega_{63}(3)$. By [12, Theorem 2.1], if Φ is a representation of S which is not the smallest representation, then $dim\Phi \geqslant (q^n - 1)(q^n - q)/(2(q + 1))$, which are 40 and 1240, respectively. But then inequality (5.3) cannot hold. If $S = S_4(7)$, then the smallest non-trivial representation in characteristic 3 of S is a Weil representation of degree 25. However, the Frobenius -Schur indicator is $\circ$, (cf. [14]), which means that $S_4(7)$ fixes no quadratic form. Thus $S_4(7)$ cannot embed in $\Omega_{25}(3)$. If Φ is a non-trivial representation of $S_4(7)$, which is not the smallest representation of S, then $dim(\Phi) \geqslant 126$, but this again violates (5.3). Thus equation (3.1) cannot hold.

(iv) Case $S \in \{S_4(2), S_6(2), S_8(2), S_4(4)\}$. By the isomorphism $S_4(2) \cong S_6$, it follows that $A_6 \cong S_4(2)'$. Thus we can exclude this case. If $S = S_4(4)$ then $dimV \geqslant 51$ and $(3^m + 1)/2 \geqslant (3^{25} + 1)/2 > |Aut(S_4(4))|$ so equation (3.1) cannot hold. For $S_6(2)$, and $S_8(2)$, we need to consider the following cases $(S_6(2), 7), (S_6(2), 21)$, $(S_6(2), 27)$ $(S_8(2), 35)$.

If $S = S_6(2)$ and $dim(V) = 7$, then by [5], $S_6(2)$ is a maximal subgroup of $\Omega_7(3)$ and it has only two orbits on $\mathfrak{E}(V)$ so that equation (3.1) holds for both types of points.

If $(S, dimV) = (S_6(2), 21)$ then $m = 10$, $Out(S) = 1$ and $S_6(2) \leqslant \Omega_{21}(3)$. There exist two non-singular vectors of different type $u_i, i = 1, 2$ which are the eigenvectors of elements of order $5, 12$, respectively and $(c_1, d_1) = (212, 165), (c_2, d_2) = (2132, 1647)$.

If $(S, dimV) = (S_6(2), 27)$ then $m = 13$ and $S_6(2) \leqslant \Omega_{27}(3)$. There exist two non-singular vectors of different type $u_i, i = 1, 2$ which are the eigenvectors of an element of order 5 and $(c_1, d_1) = (96968, 48183), (c_2, d_2) = (47912, 24663)$.

If $(S, dimV) = (S_8(2), 35)$ then $m = 17, Out(S_8(2)) = 1$ and $S_8(2) \leqslant \Omega_{35}(3)$. There exist two non-singular vectors of different type $u_i, i = 1, 2$ which are the eigenvectors of an element of order 5 and $(c_1, d_1) = (119, 0), (c_2, d_2) = (256094, 129465)$ and so equation (3.1) cannot hold.

(v) Case $S \in \{U_n(2), 3 \leqslant n \leqslant 7, U_3(4), U_3(5)\}$. As $U_3(2) \cong 3^2.Q_8$ and $U_4(2) \cong S_4(3)$, we can rule out these cases. If $S = U_5(2)$, then by [20], the smallest odd degree non-trivial 3-modular representation of S has degree 55. Thus $(3^m + 1)/2 \geqslant (3^{27}+1)/2 > |Aut(S)|$. If $S = U_3(4)$, then $dimV \in \{13, 39, 75\}$. However if $dimV \geqslant 39$ then (5.3) cannot hold, and if $dimV = 13$, then by [14], $U_3(4)$ fixes no quadratic form. If $S = U_7(2)$, then by [14], we have $dimV > 250$ and so (5.3) cannot hold. For the remaining cases, using [20], we need to consider the following cases: $(U_3(5), 21)$ and $(U_6(2), 21)$.

If $(M, dimV) = (U_3(5), 21)$ then we have $m = 10$, $Out(S) = S_3$ and $M \leqslant \Omega_{21}(3)$. For each extension M of S, there exist two non-singular vectors of different type with parameters $(c_i, d_i), i = 1, 2$, as follow: if $M = U_3(5)$ then $(c_1, d_1) = (7033, 3466)$, $(c_2, d_2) = (27145, 14854)$; if $M = U_3(5) \cdot 2$ then $(c_1, d_1) = (55437, 28562)$, $(c_2, d_2) = (55371, 28628)$.

If $(S, dimV) = (U_6(2), 21)$ then we have $m = 10$, $M \leqslant \Omega_{21}(3)$ and $Out(S) = S_3$. For each extension M of S, there exist two non-singular vectors of different type with parameters $(c_i, d_i), i = 1, 2$, as follow: if $M = U_6(2)$ then $(c_1, d_1) = (7033, 3466)$, $(c_2, d_2) = (27145, 14854)$; if $M = U_6(2) \cdot 2$ then $(c_1, d_1) = (55437, 28562)$, $(c_2, d_2) = (55371, 28628)$. We can check that equation (3.1) cannot hold in any of these cases.

(vi) Case $S = O_8^+(2)$. By [20], the smallest odd degree non-trivial 3-modular representation of S has degree 35, and the second smallest odd degree one has degree 147. Using (5.3) again, we only need to consider the 3-modular representation of $O_8^+(2)$ of degree 35. We have $m = 17$ and there are two non-singular points $v_i, i = 1, 2$, of different type with $|\langle v_1\rangle S| = |120$ and $|\langle v_2\rangle S| = 90720$. Then $|\langle v_i\rangle M| \leqslant |Out(S)||\langle v_i\rangle S| < 3^{m-1}$ and so equation (3.1) cannot hold.

(vii) Case $S = O_8^-(2)$. By [20], $dimV \geqslant 203$. Then inequality (5.3) cannot hold.

$(viii)$ Case $S = F_4(2)$. By [14], $dimV > 255$. Then inequality (5.3) cannot hold.

(ix) Case $S = G_2(4)$. By [20], $dimV \geqslant 649$. Clearly, $3^{m-1} > 4^{15} \geqslant |Aut(S)|$.

(x) Case $S = Sz(8)$. By [20], $dimV = 35$. Then inequality (5.3) cannot hold.

(xi) Case $S = {}^3D_4(2)$. By [20], either $dimV = 25$ or $dimV \geqslant 351$. If $dimV \geqslant 351$, then $m \geqslant 174$ and clearly $3^{m-1} \geqslant 3^{174} > 3 \cdot 2^{29} \geqslant |Aut(S)|$. When $dimV = 25$, the group S is not maximal in $\Omega_{25}(3)$ as ${}^3D_4(2) \leqslant F_4(3) \leqslant \Omega_{25}(3)$ (see [26]).

(xii) Case $S = {}^2F_4(2)'$. By [20], $2m + 1 \geqslant 77$. Then $3^{m-1} \geqslant 3^{37} > |Aut(S)|$. □

5.3.3. *Embedding of groups of Lie type in defining characteristic.* Let k be an algebraically closed field of characteristic p. Let $\mathfrak{G}$ be a simply connected, simple algebraic group over k. Fix a maximal torus T and a Borel subgroup B containing T. Let U be the unipotent radical of B. Then $B = UT$. Let Φ be the root system of $\mathfrak{G}$, select a system of positive roots Φ^+ from Φ, with corresponding fundamental roots $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. Let $\{\lambda_1, \dots, \lambda_\ell\}$ be the fundamental dominant weights and X^+ be the set of dominant weights. If q is a power of p, then we put $X_q = \{\sum_{i=1}^{\ell} c_i\lambda_i \mid c_i \in \mathbb{Z}, 0 \leqslant c_i \leqslant q-1\}$. A *p-restricted dominant weight* is a weight that lies in X_p. Let $\mathfrak{L}$ be the simple Lie algebra over $\mathbb{C}$ of the same type as $\mathfrak{G}$. For each dominant weight $\lambda \in X^+$, there exists an irreducible $\mathfrak{L}$-module $V(\lambda)$ of highest weight λ, and a maximal vector v^+ (unique up to scalar multiplication). Let $\mathcal{U} = \mathcal{U}(\mathfrak{L})$ be the universal enveloping algebra of $\mathfrak{L}$, and $\mathcal{U}_{\mathbb{Z}}$ be the Kostant

$\mathbb{Z}$-form of $\mathcal{U}$ (see [15, 26.3]). Now $\mathcal{U}_{\mathbb{Z}}v^+$ is the minimal admissible lattice in $V(\lambda)$, and $\mathcal{U}_{\mathbb{Z}}v^+ \otimes_{\mathbb{Z}} k$ is a $k\mathfrak{G}$-module of highest weight λ, also denoted by $V(\lambda)$, and called a *Weyl module* for $\mathfrak{G}$ ([15, 27.3]). The Weyl module $V(\lambda)$ has a unique maximal submodule $J(\lambda)$ and $L(\lambda) = V(\lambda)/J(\lambda)$ is an irreducible $k\mathfrak{G}$-module of highest weight λ. We will label the Dynkin diagram as in [28].

Assume that $\widehat{S}$ is simply connected of type A_ℓ or ${}^2A_\ell$ over $\mathbb{F}_q$, where $q = p^f$, and $\mathfrak{G}$ be the corresponding simply connected, simple algebraic group over k, such that $\widehat{S} = \mathfrak{G}_\sigma$ for some suitable Frobenius map σ. Let $N = N(\widehat{S})$ be the natural module for $\widehat{S}$. We collect here some information about $L(\lambda)$ for some special dominant weights λ.

(1) Let $0 < c < p$, and $\lambda = c\lambda_1$ or $\lambda = c\lambda_\ell$. Then $L(\lambda)$ has all weight spaces of dimension 1. $L(\lambda)$ is isomorphic to the space of homogeneous polynomials of degree c, that is, $L(\lambda) \cong S^c(N)$. In particular, $\dim L(\lambda) = \frac{(\ell+c)!}{\ell! c!}$ ([30, 1.14]).

(2) If $\ell > 1$, then $L(\lambda_i) \cong \bigwedge^i N$ and $\dim L(\lambda_i) = \binom{\ell+1}{i}$ (see [3]).

(3) Let $\lambda = n_1\lambda_1 + n_2\lambda_2 + \cdots + n_\ell\lambda_\ell$ be a dominant weight. Then $L(\lambda)$ preserves a non-degenerate bilinear form if and only if $n_1 = n_\ell, n_2 = n_{\ell-1}, \ldots$. Thus if ℓ is even then $L(\lambda_i)$ leaves invariant no non-degenerate bilinear form, and if ℓ is odd then $L(\lambda_i), i \neq \frac{\ell+1}{2}$ does not preserve any such form. Let $\lambda = \lambda_{(\ell+1)/2}$. If $\ell \equiv -1 \pmod 4$ then $L(\lambda)$ fixes a symmetric bilinear form and it fixes an alternating bilinear form if $\ell \equiv 1 \pmod 4$ (see [2] Chapter VIII §13 Table 1, p. 217).

The following constructions for adjoint modules of groups of type A_ℓ and ${}^2A_\ell$ are taken from [27], pp.491 − 492.

(4) We construct the irreducible module $L(\lambda_1 + \lambda_\ell)$ as follows: Let $V := V_1/(V_1 \cap V_2)$, where $V_1 = \{A \in M_{\ell+1}(q) \,|\, Tr(A) = 0\}, V_2 = \{aI_{\ell+1} \,|\, a \in \mathbb{F}_q\}$. Let $\widehat{S}$ act on V_1 by conjugation. Then V is an irreducible $\widehat{S}$-module of dimension $\ell^2 + 2\ell - \varepsilon_p(\ell+1)$. The bilinear form on V_1 is defined as follows: for any $A, B \in V_1$, $(A, B) = Tr(AB)$. We can check that $\widehat{S}$ preserves this bilinear form. Also, V has a basis consisting of $E_{i,j}, 1 \leqslant i < j \leqslant \ell+1, E_{ii} - E_{i+1,i+1}, i = 1, \ldots, \ell - \varepsilon_p(\ell+1)$.

(5) Let $\widehat{S} = SU_n(q)$ and $\lambda = \lambda_1 + \lambda_\ell$, where $n = \ell+1$. Let $V_2 = \{aI_n \mid a \in \mathbb{F}_{q^2}\}$, $V_1 = \{A \in M_n(q^2) \,|\, Tr(A) = 0, A = \overline{A}^t\}$, and set $V := V_1/(V_1 \cap V_2)$, where the map $A \mapsto \overline{A}$ is the map that raises each entry to its q^{th}-power. Let $\widehat{S}$ act on V_1 as in (4). The bilinear form on V_1 is also defined as in (4). We can check that $\widehat{S}$ preserves this bilinear form and $L(\lambda_1 + \lambda_\ell) \cong V$. Moreover fix a generator μ of $\mathbb{F}_{q^2}^*$, V has a basis consisting of $E_{i,j} + E_{j,i}, \mu E_{i,j} + \overline{\mu} E_{j,i} 1 \leqslant i < j \leqslant \ell+1, E_{ii} - E_{i+1,i+1}, i = 1, \ldots, \ell - \varepsilon_p(\ell+1)$.

Proposition 5.21. *Assume M is almost simple of type S, where $\widehat{S}$ is simply connected of type A_ℓ or ${}^2A_\ell$ over $\mathbb{F}_{3^f}$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1-3.*

Proof. Assume (3.1) holds for some $r \in \{s, t\}$ and any M orbits in $\mathfrak{E}_\xi(V)$. We consider the case $f = 1$ and $f > 1$ separately. We use the notation $L_n^\varepsilon(q)$ to denote $L_n(q)$ when the sign ε is $+$ and $U_n(q)$ when $\varepsilon = -$.

Case $f = 1$. We can assume that $\ell \geqslant 2$. By [16, 31.1] and [32, §13], there exists a 3-restricted dominant weight $\lambda \in X_3$ such that $V \cong L(\lambda)$. As $|Aut(S)| = |Aut(L_{\ell+1}^\varepsilon(3))| \leqslant 3^{(\ell+1)^2}$, where $\varepsilon = \pm$. It follows from (5.3) that $3^{m-1} < 3^{(\ell+1)^2}$. Hence $m < (\ell+1)^2 + 1$ and $dim V < 2(\ell+1)^2 + 3$. We need to look for all dominant weights $\lambda \in X_3$ such that $L(\lambda)$ is self-dual, has dimension less than

$2(\ell+1)^2+3$ and has odd degree. If $\ell \geqslant 18$, then $\ell^3/8 \geqslant 2(\ell+1)^2+3$, and so by [28, Theorem 5.1], λ is one of the following 3-restricted dominant weights $\{\lambda_1, \lambda_\ell, \lambda_2, \lambda_{\ell-1}, 2\lambda_1, 2\lambda_\ell, \lambda_1+\lambda_\ell\}$. Since $L(\lambda)$ is self-dual, the only possibility for λ is $\lambda_1+\lambda_\ell$. If $\ell < 18$, then by Theorem 4.4, Appendix A_6 through A_{21} in [28], either $\lambda = 2\lambda_2$ when $\ell = 3$ or $\lambda = \lambda_1+\lambda_\ell$ for $2 \leqslant \ell \leqslant 17$.

Suppose that $\ell \geqslant 4$. We have $dim L(\lambda_1+\lambda_\ell) = \ell^2+2\ell-\varepsilon_p(\ell+1)$. As $dim L(\lambda_1+\lambda_\ell)$ is odd, it follows that $\ell = 6b_1+1, 6b_1+2$ or $6b_1+3$ for $b_1 \geqslant 1$. Consequently $\ell \geqslant 7$. As constructed above, $L(\lambda_1+\lambda_\ell) \cong V := V_1/(V_1 \cap V_2)$. Let U be the subgroup of $\widehat{S}$ consisting of all matrices of the form $diag(I_2, A)$, where $A \in SL^\varepsilon_{\ell-1}(3)$. Then $U \cong SL^\varepsilon_{\ell-1}(3)$. For $\xi = \pm$, let $x_\xi = E_{1,2}+\xi E_{2,1}+V_1\cap V_2$, when $\varepsilon = +$, and $x_+ = E_{1,2}+E_{2,1}+V_1\cap V_2, x_- = \mu E_{1,2}+\overline{\mu}E_{2,1}+V_1\cap V_2$ when $\varepsilon = -$. Then $x_\xi \in V$ and $Q(x_\xi) \neq 0$. It follows that $\langle x_\xi \rangle$ is non-singular in V, of plus or minus type depending on ξ and ℓ. As $V_1 \cap V_2$ is fixed under natural action of U, and clearly, U centralizes x_ξ, it follows that $U \leqslant S_{\langle x_\xi \rangle}$, the stabilizer of $\langle x_\xi \rangle$ in S. We have $1+c+d \leqslant |Aut(S) : U| = [Aut(L^\varepsilon_{\ell+1}(3)) : SL^\varepsilon_{\ell-1}(3)] \leqslant 2\cdot 3^{2\ell-1}(3^\ell+1)(3^{\ell+1}+1) < 3^{4\ell+2}$. As $2m+1 = \ell^2+2\ell-\varepsilon_3(\ell+1) \geqslant \ell^2+2\ell-1$, $m-1 \geqslant (\ell^2+2\ell-4)/2$. We have $(\ell^2+2\ell-4)/2-(4\ell+2) = (\ell(\ell-6)-8)/2$. If $\ell \geqslant 8$ then $\ell(\ell-6)-8 > 0$, hence $m-1 > 4\ell+2$. If $\ell = 7$ then $2m+1 = \ell^2+2\ell = 63$, or $m = 31$ and $m-1 = 30 = 4\ell+2$. Thus $m-1 \geqslant 4\ell+2$ for any $\ell \geqslant 4$. Hence $3^{m-1} \geqslant 3^{4\ell+2} > 1+c+d$. This contradicts to inequality (5.3). Therefore equation (3.1) cannot hold in this case.

We are left with the cases $\ell = 2, \lambda = \lambda_1+\lambda_2$, $\ell = 3, \lambda = \lambda_1+\lambda_3$, and $\ell = 3, \lambda = 2\lambda_2$. If the first case holds then $\widehat{S} = SL^\varepsilon_3(3)$, and $dim L(\lambda_1+\lambda_2) = 7$. However by [5], $\Omega_7(3)$ has no maximal subgroup with socle $L^\varepsilon_3(3)$.

Assume that $(S, L) = (L_4(3), \Omega_{15}(3))$. For each extension of S, using [10], we can find two non-singular points of different type with parameters (c, d) as follow:

if $M = L_4(3)$ then $(c, d) = (42524, 20655)$, $(1160, 945)$;

if $M = L_4(3)\cdot 2$ then $(c, d) = (311768, 154791)$, $(505196, 252963)$.

Assume that $(S, L) = (U_4(3), \Omega_{15}(3))$. As in case $L_4(3)$, for each extension M of $U_4(3)$, we can find two non-singular points of different type with the parameters (c, d) as follow:

If $M = U_4(3)$ or $U_4(3)\cdot 2$ then $(c, d) = (435212, 217971)$, $(2780, 1755)$;

if $M = U_4(3)\cdot 2^2$ then $(c, d) = (217970, 108621)$, $(435212, 217971)$;

if $M = U_4(3)\cdot 4$ or $U_4(3)\cdot D_8$ then $(c, d) = (217970, 108621)$.

If $(S, L) = (L_4(3), \Omega_{19}(3))$ and $M = L_4(3), L_4(3)\cdot 2$ then there exist two non-singular points of different types with $(c, d) = (2600, 1611), (1070, 1035)$.

Assume $(S, L) = (U_4(3), \Omega_{19}(3))$. For each extension M of $U_4(3)$, we can find two non-singular points of different type with the parameters (c, d) as follow:

if $M = U_4(3), U_4(3)\cdot 2$ then $(c, d) = (2690, 1845), (217700, 108891)$;

if $M = U_4(3)\cdot 4$ then $(c, d) = (435752, 217431)$, $(217700, 108891)$;

if $M = U_4(3)\cdot 2^2, U_4(3)\cdot D_8$ then $(c, d) = (435752, 217431), (2420, 2115)$.

We can check that equation (3.1) cannot hold in any of these cases.

Case $f \geqslant 2$. First consider case $\ell = 1$. As $SL_2(q) \cong SU_2(q)$, we can assume that $\varepsilon = +$. If $f = 2$, then $S = SL_2(9)$. Then $\overline{S} = L_2(9) \cong A_6$. Thus, we can assume that $f \geqslant 3$. If λ is any 3-restricted dominant weight then $\lambda = c\lambda_1$, where $0 \leqslant c \leqslant 2$, $dim L(c\lambda_1) = c+1$ and $L(c\lambda_1)$ is self-dual. By [21, Theorem 5.4.5], $dim V = (dim \Psi)^f$, for some irreducible $k\widehat{S}$-module Ψ. As $dim V = 2m+1$ is odd, $dim\Psi$ is odd and hence $dim\Psi \geqslant 3 = dim L(2\lambda_1)$. It follows that $2m+1 \geqslant 3^f$ and hence $m-1 \geqslant (3^f-3)/2$. As $|Aut(L_2(3^f))| = f3^f(3^{2f}-1) < 3^{4f}$, it follows from

(5.3) that $(3^f-3)/2 \leqslant 4f$, with $f \geqslant 3$. However by induction on $f \geqslant 3$, this is not true. Thus equation (3.1) cannot hold.

Consider case $\ell \geqslant 2$. It is shown in case $f=1$ that if $\lambda \in X_3$ such that $L(\lambda)$ is self-dual and has smallest odd degree then $\lambda = \lambda_1 + \lambda_\ell$. By [21, Theorem 5.4.5] and [30, 1.11] again, $2m+1 = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ of odd degree. Hence $dim\Psi \geqslant dimL(\lambda_1+\lambda_\ell)$. It follows that $2m+1 \geqslant (\ell^2 + 2\ell - \varepsilon_3(\ell+1))^f$. We will show that $3^{m-1} > |Aut(L^{\varepsilon}_{\ell+1}(3^f))|$. Then (5.3) cannot hold. As $|Aut(L^{\varepsilon}_{\ell+1}(3^f))| < 3^{f(\ell+1)^2}$ and $m-1 \geqslant ((\ell^2+2\ell-\varepsilon_3(\ell+1))^f-3)/2 \geqslant ((\ell^2+2\ell-1)^f-3)/2$, it suffices to show that $((\ell^2+2\ell-1)^f-3)/2 > f(\ell+1)^2$. This is true by induction. The proof is complete. □

Let $\widehat{S}$ be a simply connected group of type B_ℓ over $\mathbb{F}_q$, where $q=p^f$, and $\mathfrak{G}$ be the corresponding simply connected, simple algebraic group over k, such that $S = \mathfrak{G}_\sigma$ for some suitable Frobenius map σ. Let N be the natural module for $\widehat{S}$ with the standard basis $\beta = \{e_1, \ldots, e_\ell, x, f_1, \ldots, f_\ell\}$. Multiplying some suitable constant to the symmetric bilinear form, we can assume that the representing matrix of the symmetric bilinear form on N has the form

$$B = \begin{pmatrix} 0 & 0 & I_\ell \\ 0 & 1 & 0 \\ I_\ell & 0 & 0 \end{pmatrix}.$$

Let T be the set of all matrices of the form $diag(d,1,d^{-1})$, $d = diag(t_1, \ldots, t_\ell)$ $\in GL_\ell(k)$. As $T \cong (k^*)^\ell$, T is a maximal torus of $\widehat{S}$. For $i = 1, \ldots, \ell$, define $\gamma_i : T \to k^*$, by $\gamma_i(d,1,d^{-1})) = t_i$. Then $\{\gamma_i\}_{i=1}^{\ell}$ form an orthonormal basis for E. Also define $\alpha_{\ell+1-i} = \gamma_{\ell+1-i} - \gamma_{\ell-i}$, for $i = 1, \ldots, \ell-1$, and $\alpha_1 = \gamma_1$. Then $\{\alpha_1, \ldots, \alpha_\ell\}$ is a fundamental root system of type B_ℓ, and the corresponding $\mathbb{Z}$-basis of the fundamental dominant weights is $\{\lambda_1, \ldots, \lambda_\ell\}$, defined as following, for $i = 1\ldots, \ell-1$ $\lambda_1 = \frac{1}{2}(\gamma_1 + s + \gamma_\ell)$, and $\lambda_{\ell+1-i} = \gamma_\ell + \gamma_{\ell-1} + \cdots + \gamma_{\ell+1-i}$,

Proposition 5.22. *Assume M is almost simple of type S, where $\widehat{S}$ is simply connected of type B_ℓ over $\mathbb{F}_{3^f}$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1-3.*

Proof. Assume (3.1) holds for some $r \in \{s,t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$.

Case $f=1$**.** First we claim that if λ is a 3-restricted dominant weight such that $dimL(\lambda)$ is odd and greater than $dimN$ then λ must be one of the following weights:

(*i*) $\lambda = \lambda_{\ell-1}, \ell \geqslant 3, \ell$ odd, and $dimL(\lambda) = 2\ell^2 + \ell$;

(*ii*) $\lambda = 2\lambda_\ell, \ell = 6k+3, 6k+4$ or $6k+5$, for some non-negative integer k, and $dimL(\lambda) = 2\ell^2+3\ell, 2\ell^2+3\ell-1, 2\ell^2+3\ell$, respectively;

(*iii*) $\ell = 3, \lambda = 2\lambda_1$, and $dimL(\lambda) = 35$.

From (5.3), we have $3^{m-1} \leqslant |Aut(S)| = |Aut(\Omega_{2\ell+1}(3))| = 3^{\ell^2}\prod_{i=1}^{\ell}(3^{2i}-1) \leqslant 3^{2\ell^2+\ell}$. Hence $dimL(\lambda) = 2m+1 \leqslant 4\ell^2+2\ell+3$. Notice that if $\ell \geqslant 5$ then $\ell^3 - (4\ell^2+2\ell+3) \geqslant 5\ell^2 - 4\ell^2 - 2\ell - 3 = (\ell-1)^2 - 4 > 0$, and so $\ell^3 > 4\ell^2+2\ell+3$. If $\ell > 11$, then $dimL(\lambda) \leqslant 4\ell^2+2\ell+3 < \ell^3$, and hence by [28, Theorem 5.1], λ is either $\lambda_{\ell-1}$ or $2\lambda_\ell$. For $2 \leqslant \ell \leqslant 11$, by [28, Theorem 4.4] and the upper bound for dimension of $L(\lambda)$ above, again, λ is one of the weights above or $\ell = 3$, $\lambda = 2\lambda_1$, and $dimL(2\lambda_1) = 35$. So case (*iii*) holds. It remains to get the restriction on ℓ in cases (*i*) and (*ii*). From the reference above, we also have $dimL(\lambda_{\ell-1}) = \ell(2\ell+1)$

and $dimL(2\lambda_\ell) = 2\ell^2 + 3\ell - \varepsilon_3(2\ell+1)$. Now case (i) holds as $dimL(\lambda_{\ell-1})$ is odd if and only if ℓ is odd. Suppose that $3 \mid 2\ell+1$. then as $2\ell+1$ is odd, $2\ell+1 = 3(2t+1)$, hence $\ell = 3t+1$. Since $dimL(2\lambda_\ell) = 2\ell^2 + 3\ell - 1 = \ell(2\ell+3) - 1$ is odd, it follows that $\ell = 3t+1$ is even. Thus $t = 2k+1$ and $\ell = 6k+4$. With the same argument, we can see that if $3 \nmid 2\ell+1$, then $\ell = 6k+3$ or $6k+4$.

Let Q be the non-degenerate quadratic form associated with the non-degenerate symmetric bilinear form on N. Then for $v \in N, (v,v) = 2Q(v)$. On the tensor product $N \otimes N$, we can define a non-degenerate symmetric bilinear form induced from the form on N as follows: for $u_i, w_i \in N, i = 1,2$, $(u_1 \otimes u_2, w_1 \otimes w_2) = (u_1, w_1)(u_2, w_2)$, and extend linearly on $N \otimes N$. Recall that if u is a non-singular vector in N, then the reflection $r_u : N \to N$ is defined by $vr_u = v - \frac{(v,u)}{Q(u)}u$, for any $v \in N$. Let $\widehat{S}$ act on $N \otimes N$ by $(u \otimes v)g = (ug \otimes vg)$. Then $((u_1 \otimes u_2)r_u, (w_1 \otimes w_2)r_u) = ((u_1 \otimes u_2, w_1 \otimes w_2)$ for any non-singular vector u. Thus, $\wedge^2 N$ and $S^2(N)$ leave invariant symmetric bilinear forms induced from the one on $N \otimes N$. We have $L(\lambda_{\ell-1}) \cong \wedge^2(N)$ and $L(2\lambda_\ell) \cong w^\perp/(w^\perp \cap \langle w \rangle)$, where $w = \sum_{i=1}^{\ell}(e_i \otimes f_i + f_i \otimes e_i) + x \otimes x \in S^2(N)$.

We now consider case (i). As $L(\lambda_{\ell-1}) \cong \wedge^2 N$, $dimL(\lambda_{\ell-1}) = \ell(2\ell+1)$ and $L(\lambda_{\ell-1})$ has a basis consisting of $e_i \wedge e_j, f_i \wedge f_j, 1 \leqslant i < j \leqslant \ell$, $e_i \wedge f_j, 1 \leqslant i,j \leqslant \ell$ and $e_i \wedge x, x \wedge f_i, 1 \leqslant i \leqslant \ell$. Also denote by Q the associated quadratic form on $N \otimes N$. Then for $\xi \in \{\pm 1\}$, let $v = e_1 \wedge x + \xi x \wedge f_1 = (e_1 - \xi f_1) \wedge x$. Since $Q(e_1 \wedge x) = 0 = Q(x \wedge f_1)$, we have $Q(v) = (e_1 \wedge x, x \wedge f_1) = \xi$. Hence v is a non-singular point. Let N_1 be the subspace of N generated by $\{e_1 - \xi f_1, x\}$. As N_1 is non-degenerate, $N = N_1 \perp N_1^\perp$. Denote by H the centralizer of N_1 in $\Omega(N) \cong \Omega_{2\ell+1}(3)$. It follows that $H \cong \Omega_{2\ell-1}(3)$, and H fixes v. By (5.3) we have $3^{m-1} \leqslant 1 + c + d \leqslant |Aut(\Omega_{2\ell+1}(3)) : \Omega_{2\ell-1}(3)| = 2 \cdot 3^{2\ell-1}(3^{2\ell} - 1) \leqslant 3^{4\ell}$. Hence $m - 1 < 4\ell$, so that $2\ell^2 + \ell = 2m + 1 < 8\ell + 3$. As ℓ is odd and $\ell > 1$, the above inequality holds only when $\ell = 3$. In this case, we have $\Omega_7(3) \leqslant \Omega_{27}(3)$. Using [10], there are two non-singular points of different type $\langle x_i \rangle, i = 1,2$, with $(c_i, d_i) = (13040, 9072), (26324, 17901)$, we see that equation (3.1) cannot hold in this case.

In case (ii), for $\xi \in \{\pm 1\}$, let $v = e_1 \otimes e_1 + \xi f_1 \otimes f_1 + \langle w \rangle \cap w^\perp$. Then $Q(v) = \xi$, hence v is non-singular in $L(2\lambda_\ell)$. Let $N_1 = \langle e_1, f_1 \rangle$. Then N_1 is a non-degenerate subspace of N. As in case (i), let H be the centralizer of N_1 in $\Omega(N)$, as $H \cong \Omega_{2\ell-1}(3)$, we have $3^{m-1} \leqslant 1 + c + d \leqslant |Aut(\Omega_{2\ell+1}(3)) : \Omega_{2\ell-1}(3)| < 3^{4\ell}$, hence $2m+1 < 8\ell+3$. As $dimL(2\lambda_\ell) = 2\ell^2 + 3\ell - \varepsilon_3(2\ell+1)$, it follows that $2\ell^2 + 3\ell - 1 \leqslant 2\ell^2 + 3\ell - \varepsilon_3(2\ell+1) < 8\ell+3$. If $\ell \geqslant 4$ then $2\ell^2 + 3\ell - 1 \geqslant 24\ell + 3\ell - 1 = 8\ell + (3\ell - 1) > 8\ell + 11 > 8\ell + 3$, and if $\ell = 3$, then $2\ell^2 + 3\ell - \varepsilon_3(2\ell+1) = 27 = 8\ell + 3$. Since $\ell \geqslant 3$ in this case, (5.3) cannot hold.

Finally $\ell = 3$ and $\lambda = 2\lambda_1$. In this case, we have $\Omega_7(3) \leqslant \Omega_{27}(3)$. Using [10], there are two non-singular points of different type $\langle x_i \rangle, i = 1,2$, with $(c_i, d_i) = (13850, 8262), (26324, 17901)$, we see that equation (3.1) cannot hold in this case.

Case $f \geqslant 2$. By [21, Theorem 5.4.5] and [30, 1.11], $dimV = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ. As $dimV$ is odd, so is $dim\Psi$. Firstly, suppose that $f \geqslant 3$. Since $dim\Psi$ is at least $2\ell+1$, it follows that $2m+1 \geqslant (2\ell+1)^f$. By (5.3), we have $3^{m-1} \leqslant |Aut(\Omega_{2\ell+1}(3^f))| < f \cdot 3^{f(2\ell^2+\ell)} \leqslant 3^{f(2\ell^2+\ell+1)}$. As $2m+1 \geqslant (2\ell+1)^f$, we have $\frac{1}{2}((2\ell+1)^f - 3) < f(2\ell^2 + \ell + 1)$. Clearing fraction, we get $(2\ell+1)^f - 3 - f(4\ell^2 + 2\ell + 2) < 0$. By induction, this inequality cannot happen.

Secondly, suppose $f = 2$ and $dim\Psi > 2\ell + 1$. It follows from case $f = 1$ that $dim\Psi \geqslant 2\ell^2 + \ell$. Arguing as above, we have $(\ell(2\ell+1))^2 - 3 < 2 \cdot 2(2\ell^2 + \ell + 1) = 2(4\ell^2+2\ell+2)$. As $\ell \geqslant 2$, $\ell^f(2\ell+1)^f - 3 - 2(4\ell^2+2\ell+2) \geqslant 2^2(2\ell+1)^2 - 2(2\ell+1)^2 + 4\ell - 5 \geqslant 2(2\ell+1)^2 + 3 > 0$. Hence $\ell^f(2\ell+1)^f - 3 > 2(4\ell+2\ell+2)$, a contradiction.

Finally, suppose $f = 2$ and $dim\Psi = 2\ell + 1$. In this case, we can assume that $\Psi \cong L(\lambda_\ell) \cong N$, and so $V = N \otimes N^{(1)}$, where N is the natural module for $\Omega_{2\ell+1}(3^2)$, and $N^{(1)}$ denote the module received from the twist action of $\widehat{S}$ on N. For any element $v \in N$, denote by $v^{(1)}$ the corresponding element in $N^{(1)}$. Notice that if p is odd then for any $a, b \in \mathbb{F}_p^f$, $a + b = 0$ or 1 if and only if $a^p + b^p = 0$ or 1, correspondingly. This holds because $a^p + b^p = (a + b)^p$. Fix a standard basis $\beta = \{e_1, \ldots, e_\ell, x, f_\ell, \ldots, f_1\}$ of S. Let $u = (\varepsilon_1 + \xi f_1) \in N$. Then for $g \in S$, in the basis β, we write $g = (a_{i,j})$. Assume that $ug = g$. Then $a_{11} + a_{2\ell+1,1} = 1 = a_{1,2\ell+1} + a_{2\ell+1,2\ell+1}$ and $a_{1i} + a_{2\ell+1,i} = 0, 1 < i < 2\ell + 1$. Hence, by the notice above, we have $a_{11}^p + a_{2\ell+1,1}^p = 1 = a_{1,2\ell+1}^p + a_{2\ell+1,2\ell+1}^p$ and $a_{1i}^p + a_{2\ell+1,i}^p = 0, 1 < i < 2\ell + 1$. Therefore, $u^{(1)}g = ug^\nu = u(a_{ij}^p) = u = u^{(1)}$. This means that if $g \in \widehat{S}$ fixes u then g also fixes $u^{(1)}$. Let $v = u \otimes u^{(1)} \in N \otimes N^{(1)}$. Let H be the stabilizer of u in $\widehat{S}$. Then $H \cong \Omega_{2\ell}^\varepsilon(3^2)$ with $\varepsilon = \pm 1$, and $H \leqslant \widehat{S}_v$. Hence by (5.3), $3^{m-1} \leqslant |Aut(\Omega_{2\ell+1}(3^2)) : \Omega_{2\ell}^\varepsilon(3^2)| \leqslant |Aut(\Omega_{2\ell+1}(3^2)) : \Omega_{2\ell}^+(3^2)| \leqslant 3^{4\ell+2}$. Since $2m + 1 = (2\ell + 1)^2$, it follows that $(2\ell + 1)^2 < 8\ell + 7$. As $\ell \geqslant 2$, $(2\ell+1)^2 - 8\ell - 7 = 4\ell^2 + 4\ell + 1 - 8\ell - 7 = 4\ell(\ell-1) - 6 > 0$. This final contradiction finishes the proof. □

Let $\widehat{S}$ be a simply connected group of type C_ℓ over $\mathbb{F}_q$, where $q = p^f$, and $\mathfrak{G}$ be the corresponding simply connected, simple algebraic group over k, such that $\widehat{S} = \mathfrak{G}_\sigma$ for some suitable Frobenius map σ. Let N be the natural module for $\widehat{S}$ with the standard basis $\beta = \{e_1, \ldots, e_\ell, f_1, \ldots, f_\ell\}$. The representing matrix of the non-degenerate symplectic form on N has the form

$$B = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}.$$

From the isomorphisms $Sp_2(q) \cong SL_2(q)$, and $Sp_4(q) \cong \Omega_5(q)$ for q odd, we can assume that $\ell \geqslant 3$.

Proposition 5.23. *Assume M is almost simple of type S, where $\widehat{S}$ is simply connected of type C_ℓ over $\mathbb{F}_{3^f}$, with $\ell \geqslant 3$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $(\ell, \lambda, dimV) = (3, \lambda_2, 13)$ or $(L, S, \lambda) = (\Omega_{41}(3), PSp_8(3), \lambda_1)$. If the first case holds then M has at most two orbits on $\mathfrak{E}_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7 and hence $(L, S) = (\Omega_{13}(3), PSp_6(3))$ is in Table 2. The last case is in Table 3.*

Proof. Assume (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$.

Case $f = 1$. Let $\lambda \in X_3$ be a 3-restricted dominant weight such that $L(\lambda) \cong V$. We first get an upper bound for $dimV$. As $|Aut(PSp_{2\ell}(3))| = f 3^{\ell^2} \prod_{i=1}^{\ell}(3^{2i} - 1) \leqslant 3^{2\ell^2+\ell}$, by (5.3), $3^{m-1} \leqslant 3^{2\ell^2+\ell}$, and hence $2m + 1 \leqslant 4\ell^2 + 2\ell + 3$. If $\ell \geqslant 5$ then $4\ell^2 + 2\ell + 3 < \ell^3$, and so $dimV < \ell^3$. If $\ell > 11$, then $dimL(\lambda) < \ell^3$, and hence by [28, Theorem 5.1], λ is either $\lambda_{\ell-1}$ or $2\lambda_\ell$. For $2 \leqslant \ell \leqslant 11$, by [28, Theorem 4.4] and the upper bound for dimension of $L(\lambda)$ above, again, λ is one of the weights above or $\ell = 4$, $\lambda = \lambda_1$, and $dimL(\lambda_1) = 41$. We have $dimL(2\lambda_\ell) = \ell(2\ell + 1), L(2\lambda_\ell) \cong S^2(N)$, and $dimL(\lambda_{\ell-1}) = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell), L(\lambda_{\ell-1}) \cong w^\perp/(\langle w\rangle \cap w^\perp)$, where

$w = e_1 \wedge f_1 + s + e_\ell \wedge f_\ell$. In these cases, $\widehat{S}$ leaves invariant a quadratic form Q induced from the symplectic form on N. In case $\lambda = 2\lambda_\ell$, let $v = e_1 \otimes e_1 + \xi f_1 \otimes f_1 \in L(2\lambda_\ell)$. Since $dim L(2\lambda_\ell) = \ell(2\ell+1)$ is odd, ℓ must be odd.

If $\ell = 3$ then $dim L(2\lambda_\ell) = 21$ and $PSp_6(3) \leqslant \Omega_{21}(3)$. Using [10], there are two non-singular points of different type $\langle x_i \rangle, i = 1, 2$, with $(c_i, d_i) = (7075430, 3538809)$, $(26324, 17901)$, we see that equation (3.1) cannot hold in this case.

Thus we assume that $\ell \geqslant 5$. Let H be the centralizer in $\widehat{S}$ of the subspace generated by $\{e_1, f_1\}$. Then $H \cong Sp_{2\ell-2}(3)$. By (5.3), we have $3^{m-1} \leqslant |Aut(PSp_{2\ell}(3)) : PSp_{2\ell-2}(3)| = 2 \cdot 3^{\ell^2}(3^{2\ell}-1)/3^{(\ell-1)^2} < 3^{4\ell}$. Hence $2m+1 < 8\ell+3$. As $2m+1 = \ell(2\ell+1)$, it follows that $\ell(2\ell+1) < 8\ell+3$. However as $\ell \geqslant 5$, $\ell(2\ell+1) \geqslant 5(2\ell+1) = 10\ell+5 > 8\ell+3$, a contradiction. Next consider case $\lambda = \lambda_{\ell-1}$. As $dim L(\lambda_{\ell-1}) = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell)$ is odd, $\ell = 6k+2, 6k+3$ or $6k+4$.

Let $v = e_1 \wedge e_2 + \xi f_1 \wedge f_2 + \langle w \rangle \cap w^\perp$, where $\xi = \pm 1$. Then v is non-singular in V. Let $N_1 = \langle e_1, e_2, f_1, f_2 \rangle$ be a subspace of N. Since N_1 is non-degenerate, $N = N_1 \perp N_1^\perp$. Let H, K be the centralizers in $\widehat{S}$ of $N_1, N_1^\perp$, respectively. Then $H \cong Sp_{2(\ell-2)}(3)$, $K \cong Sp_4(3)$, and H, K commute. In the basis $\beta_1 = \{e_1, e_2, f_1, f_2\}$, let

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\xi & 0 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

As $gBg^t = B, hBh^t = B$, and $det(g) = 1 = det(h)$, $g, h \in Sp(V_1)$. Furthermore, g, h are of order 3 and $gh = hg$, the subgroup generated by g and h are elementary abelian of order 9. Since $vg = v$ and $vh = h$, it follows that $E = \langle g, h \rangle \leqslant K_v$. Thus $E \times H \leqslant \widehat{S}_v$, hence $1 + c + d \leqslant |Aut(S) : (E \times H)| = |Aut(PSp_{2\ell}(3)) : (E \times PSp_{2(\ell-2)}(3))| < 3^{8\ell-7}$. Hence $2m+1 < 16\ell - 11$. Since $2m+1 = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell) \geqslant 2\ell^2 - \ell - 2$, we have $2\ell^2 - \ell - 2 < 16\ell - 11$, or equivalent $2\ell^2 - 17\ell + 9 < 0$. As $\ell = 6k+2, 6k+3, 6k+4$, if $\ell > 4$ then $k \geqslant 1$, and so $\ell \geqslant 8$. Then $2\ell^2 - 17\ell + 9 \geqslant 2\ell^2 - 16\ell - (\ell - 8) + 1 = (2\ell - 1)(\ell - 8) + 1 > 0$, a contradiction. Thus $\ell \leqslant 4$.

If $\ell = 4$, then $dim L(\lambda_{\ell-1}) = 27$, by using [10], equation (3.1) cannot hold. If $\ell = 3$, then $dim L(\lambda_{\ell-1}) = 13$. Using [10], there is only one orbit of minus points and two orbits of plus points. Hence equation (3.1) holds for both types of points by Corollary 3.7. We are left with case $\ell = 4$, $\lambda = \lambda_1, dim L(\lambda_1) = 41$.

Case $f \geqslant 2$. It follows from case $f = 1$ that if λ is a 3-restricted dominant weight such that $L(\lambda)$ is self-dual and is of odd degree then $dim L(\lambda) \geqslant 2\ell^2 - \ell - 1 - \varepsilon_p(\ell) \geqslant 2\ell^2 - \ell - 2$. By [21, Theorem 5.4.5] and [30, 1.11], $2m+1 = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ of odd degree. Thus $2m+1 = (dim\Psi)^f \geqslant (2\ell^2 - \ell - 2)^f$. By (5.3), $3^{m-1} \leqslant |Aut(PSp_{2\ell}(3))| \leqslant 3^{f(2\ell^2+\ell+1)}$, so that $2m+1 < 2f(2\ell^2 + \ell + 1) + 3$. Combining these inequalities, we have $(2\ell^2 - \ell - 2)^f < 2f(2\ell^2 + \ell + 1) + 3$, where $\ell \geqslant 3, f \geqslant 2$. However by induction this inequality cannot happen. □

Let $\widehat{S}$ be a simply connected group of type D_ℓ or ${}^2D_\ell$ over $\mathbb{F}_q$, where $q = p^f$, and $\mathfrak{G}$ be the corresponding simply connected, simple algebraic group over k, such that $S = \mathfrak{G}_\sigma$ for some suitable Frobenius map σ. Let N be the natural module for $\widehat{S}$ with the standard basis β. The representing matrix of the non-degenerate symmetric bilinear form on N has the form

$$B = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}.$$

From the isomorphisms $\Omega_4^+(q) \cong SL_2(q) \circ SL_2(q), \Omega_4^-(q) \cong L_2(q)$ and $P\Omega_6^{\pm}(q) \cong L_4^{\pm}(q)$, we can assume that $\ell \geqslant 4$.

Proposition 5.24. *Assume M is almost simple of type S, where $\widehat{S}$ is simply connected of type D_ℓ or ${}^2D_\ell$ over $\mathbb{F}_{3^f}$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1-3.*

Proof. Assume (3.1) holds for some $r \in \{s,t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$.

Case $f=1$. Let $\lambda \in X_3$ be a 3-restricted dominant weight such that $L(\lambda) \cong V$. By inequality (5.3), $3^{m-1} \leqslant |Aut(P\Omega_{2\ell}^{\varepsilon}(3))| \leqslant 3^{2\ell^2-\ell+2}$. Thus $2m+1 \leqslant 4\ell^2-2\ell+7$. By [28, Theorem 5.1 and 4.4], either $\ell \geqslant 4$ and $\lambda = \lambda_{\ell-1}, 2\lambda_\ell$ or $\ell = 4$ and $\lambda = 2\lambda_1, 2\lambda_2$.

If $\lambda = \lambda_{\ell-1}$, then $dimL(\lambda) = 2\ell^2-\ell$ and $L(\lambda) \cong \wedge^2 N$. Since $dimL(\lambda)$ is odd, ℓ must be odd. Thus $\ell \geqslant 5$. Let $a = e_1 - \xi f_1, b = e_2 + f_2 \in N$ and $z = a \wedge b \in \wedge^2 N$, where $\xi = \pm 1$. Then $\langle z \rangle$ is non-singular in $V = \wedge^2 N$. Also, let N_1 be a subspace of N generated by a and b. Clearly, N_1 is non-degenerate, $dim(N_1) = 2$, and $sgn(N_1) = \xi$. Since $N = N_1 \perp N_1^{\perp}, dim(N_1^{\perp}) = 2\ell - 2$ and $sgn(N) = sgn(N_1) \cdot sgn(N_1^{\perp})$, it follows from [21, Proposition 2.5.11] that $sgnN_1^{\perp} = \varepsilon\xi$. Since the discriminant $D(Q) \equiv detB = (-1)^\ell = -1 \pmod{(\mathbb{F}^*)^2}$, as ℓ is odd, by [21, Proposition 4.1.6], $H = \Omega(N_1^{\perp}) \cong \Omega_{2\ell-2}^{\varepsilon\xi}(3) \leqslant \Omega(N)$ centralizes N_1. Hence $H \leqslant M_{\langle z \rangle}$. Therefore $1+c+d \leqslant |Aut(P\Omega_{2\ell}^{\varepsilon}(3)) : \Omega_{2\ell-2}^{\varepsilon\xi}(3)| = 4 \cdot 3^{2\ell-2}(3^\ell - \varepsilon)(3^{\ell-1}+\varepsilon\xi) \leqslant 3^{4\ell-1}$. Since $\ell \geqslant 5, 3^{m-1} = 3^{(2\ell^2-\ell-3)/2} > 3^{4\ell-1} > 1+c+d$, a contradiction to (5.3).

If $\lambda = 2\lambda_\ell$, then $dimL(\lambda) = 2\ell^2+\ell-1-\varepsilon_3(\ell)$ and $V = L(\lambda) \cong w^{\perp}/(w^{\perp} \cap \langle w \rangle)$, where $w = \sum_{i=1}^{\ell}(e_i \otimes f_i + f_i \otimes e_i) \in S^2 N$ if $\widehat{S}$ is of type D_ℓ and $w = \sum_{i=1}^{\ell-1}(e_i \otimes f_i + f_i \otimes e_i) + x \otimes x + y \otimes y$ otherwise. Let $z_\xi = e_1 \otimes e_1 + \xi f_1 \otimes f_1 + \langle w \rangle \cap w^{\perp}, \xi = \pm 1$, and $N_1 = \langle e_1, f_1 \rangle \leqslant N$. Then z_ξ is non-singular in V. Let $N_1 = \langle e_1, f_1 \rangle \leqslant N$ and H be the centralizer of N_1 in $\Omega(N) \cong \Omega_{2\ell}^{\varepsilon}(3)$. Since $sgn(N_1^{\perp}) = \varepsilon$, $H \cong \Omega_{2\ell-2}^{\varepsilon}(3)$ and H fixes $\langle z_\xi \rangle$. Thus $1+c+d \leqslant |Aut(P\Omega_{2\ell}^{\varepsilon}(3)) : \Omega_{2\ell-2}^{\varepsilon}(3)| \leqslant 12 \cdot 3^{2\ell-2}(3^\ell - \varepsilon)(3^{\ell-1}+\varepsilon) \leqslant 3^{4\ell}$. If $\ell \geqslant 5$, then $3^{m-1} \geqslant 3^{(2\ell^2+\ell-5)/2} > 3^{4\ell} > 1+c+d$, contradicts to (5.3). If $\ell = 4$, then $2m+1 = 35$, hence $m-1 = 16$. As $3^{m-1} = 3^{16} = 3^{4\ell} > 1+c+d$, we also get a contradiction to (5.3).

Case $f \geqslant 2$. It follows from case $f = 1$ that if λ is a 3-restricted dominant weight such that $L(\lambda)$ is self-dual and is of odd degree then $dimL(\lambda) \geqslant 2\ell^2 - \ell$. As V is an absolutely irreducible $k\widehat{S}$-module, by [21, Theorem 5.4.5] and [30, 1.11], $2m+1 = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ of odd degree. Thus $2m+1 = (dim\Psi)^f \geqslant (2\ell^2-\ell)^f$. By (5.3), $3^{m-1} \leqslant |Aut(P\Omega_{2\ell}^{\varepsilon}(3))| \leqslant 3 \cdot 3^{f(2\ell^2-\ell+1)}$, so that $2m+1 < 2f(2\ell^2-\ell+1)+5$. Combining these two inequalities, we have $(2\ell^2-\ell)^f < 2f(2\ell^2-\ell+1)+5$, where $\ell \geqslant 4, f \geqslant 2$. By induction, this cannot happen. □

Assume that $\widehat{S}$ is simply connected of exceptional type defined over a field of characteristic 3. Then $\widehat{S}$ is one of the following types: $G_2, F_4, E_6, E_7, E_8, {}^2E_6, {}^3D_4, {}^2G_2$.

Proposition 5.25. *Assume M is almost simple of type S, where $\widehat{S}$ is of exceptional type above and is defined over $\mathbb{F}_{3^e}$ with $e \geqslant 1$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $(L,S,\lambda) = (\Omega_7(3), G_2(3), \lambda_i)$, $i = 1,2$, or $(F_4(3), \Omega_{25}(3), \lambda_4)$, and M has one or at most two orbits on $\mathfrak{E}(V)$, respectively, so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7 and so they are in Table 2 or $(L,S,\lambda) = (\Omega_{77}(3), E_6(3), \lambda_2)$, $(\Omega_{133}, E_7(3), \lambda_1)$ and they are in Table 3.*

Proof. Assume (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$.

(a) **Case** G_2. By (5.3) $3^{m-1} \leqslant |Aut(G_2(3^e))| \leqslant 3^{15e}$. Thus $2m+1 \leqslant 30e+3$. Assume first that $e = 1$. Then $2m+1 \leqslant 33$. Let $\lambda \in X_3$ be a 3-restricted dominant weight with $V \cong L(\lambda)$. From Appendix A.49 in [28], λ is one of the following weights $\lambda_1, \lambda_2, 2\lambda_1, 2\lambda_2$. If λ is $\lambda_i, i = 1, 2$, then $dimV = 7$. In these cases, we have $G_2(3) \leqslant \Omega_7(3)$. By [5], these are maximal embeddings and $G_2(3)$ has only one orbit in $\mathfrak{E}_\xi(V)$. If $\lambda = 2\lambda_1$ or $2\lambda_2$ then $dimV = 27$. We have $G_2(3) \leqslant \Omega_{27}(3)$. But this is not a maximal embedding as $G_2(3) \leqslant \Omega_7(3) \leqslant \Omega_{27}(3)$, where the last embedding arises from the symmetric square of the natural module for $\Omega_7(3)$ (see Proposition 5.22). Assume that $e \geqslant 2$. By [21, Theorem 5.4.5] and [30, 1.11], $2m+1 = (dim\Psi)^e$, for some self-dual irreducible $k\widehat{S}$-module Ψ of odd degree. It follows that $2m+1 \geqslant 7^e$. Combining with $2m+1 \leqslant 30e+3$, we have $e = 2$. Then $dimV = 7^2 = 49$. However $G_2(9)$ is not maximal in $\Omega_{49}(3)$ since $G_2(9) \leqslant \Omega_7(9) \leqslant \Omega_{7^2}(3)$ where the first embedding arises as in previous case, while the second is the twisted tensor product embedding.

(b) **Case** F_4. By (5.3), $3^{m-1} \leqslant |Aut(F_4(3^e))| \leqslant 3^{53e}$. So $2m+1 \leqslant 106e+3$. Assume that $e = 1$. From Appendix A.50 in [28], $\lambda = \lambda_4$ and $dimL(\lambda) = 25$. In this case, we have an embedding $F_4(3) \leqslant \Omega_{25}(3)$. By [4], $F_4(3)$ has 5 orbits of points in V. But there are two orbits of singular points, and so there are at most two orbits for each types of non-singular points. Assume that $e \geqslant 2$. we have $2m+1 \geqslant 25^e$, so that $25^e \leqslant 106e+3$. But this cannot happen for any $e \geqslant 2$.

(c) **Case** ${}^\varepsilon E_6$. By (5.3), $3^{m-1} \leqslant |Aut({}^\varepsilon E_6(3^e))| \leqslant 3^{79e}$. Thus $2m+1 \leqslant 158e+3$. Assume that $e = 1$. From Appendix A.51 in [28], $\lambda = \lambda_2$ and $dimL(\lambda) = 77$. (Note that $dimL(\lambda_1) = dimL(\lambda_6) = 27$ but these modules are not self-dual). In this case, we have ${}^2E_6(3) \leqslant E_6(3) \leqslant \Omega_{77}(3)$, and $V = L(E_6)/Z(L(E_6))$, where $L(E_6)$ is the Lie algebra of E_6 over $\mathbb{F}_3$. If $e \geqslant 2$, then $77^e \leqslant 158e+3$. However this cannot happen for any $e \geqslant 2$..

(d) **Case** E_7. By (5.3), $3^{m-1} \leqslant |Aut(E_7(3^e))| \leqslant 3^{134e}$. Thus $2m+1 \leqslant 268e+3$. Assume that $e = 1$. From Appendix A.52 in [28], $\lambda = \lambda_1$ and $dimL(\lambda) = 133$. We have $E_7(3) \leqslant \Omega_{133}(3)$, and $V = L(E_7)$, the Lie algebra of E_7 over $\mathbb{F}_3$. Assume that $e \geqslant 2$. We have $133^e \leqslant 268e+3$. This cannot happen for $e \geqslant 2$.

(e) **Case** E_8. By (5.3), $3^{m-1} \leqslant |Aut(E_8(3^e))| \leqslant 3^{249e}$. Thus $2m+1 \leqslant 498e+3$. Assume that $e = 1$. From Appendix A.53 in [28], $dimL(\lambda) \geqslant 3875 > 498.1{+}3 = 501$. Assume that $e \geqslant 2$. Clearly $3875^e > 498e+3$ for any $e \geqslant 2$.

(f) **Case** ${}^3D_4$. By (5.3), $3^{m-1} \leqslant |Aut({}^3D_4(3^e))| \leqslant 3^{30e}$. Thus $2m+1 \leqslant 60e+3$. Assume that $e = 1$. From Appendix A.53 in [28], $\lambda = 2\lambda_1, 2\lambda_2, 2\lambda_4$ and $dimL(\lambda) = 35$. Since the splitting field for ${}^3D_4(3)$ is $\mathbb{F}_{3^3}$, $dimV = 3 \cdot 35 = 105 > 63$. Assume that $e \geqslant 2$. Clearly $35^e > 60e+3$ for any $e \geqslant 2$.

(g) **Case** ${}^2G_2$. By (5.3), $3^{m-1} \leqslant |Aut({}^2G_2(3^{2e+1}))| \leqslant 3^{8(2e+1)}$. Thus $2m+1 \leqslant 32e+19$. By [21, Theorem 5.4.5], $2m+1 \geqslant 7^{2e+1}$, and so $7^{2e+1} \leqslant 32e+19$. This cannot happen. □

5.3.4. *Embedding of Sporadic groups.* Let S be a sporadic simple group and let $\widehat{S}$ be the universal covering group of S. Define $g_3(S) := [2log_3(|Aut(S)|)+4]$ and $\Re_3(S)$ to be the minimal degrees of irreducible faithful representations of S and its covering groups over $\mathbb{F}_3$.

Lemma 5.26. *Let S be a simple sporadic group. Then $\Re_3(S)$ and $g_3(S)$ are given in Table 7.*

Table 7. Minimal degrees of representations for sporadic groups in characteristic 3.

S	$\Re_3(S)$	$g_3(S)$	S	$\Re_3(S)$	$g_3(S)$
M_{11}	5	20	Suz	64	54
M_{12}	10	26	$2.Suz$	12	54
$2.M_{12}$	6	26	$O'N$	154	54
J_1	56	25	$Co3$	22	53
M_{22}	21	28	$Co2$	23	61
$2.M_{22}$	10	28	Fi_{22}	77	63
J_2	13	29	$2.Fi_{22}$	176	63
$2.J_2$	6	29	HN	133	65
M_{23}	22	33	Ly	651	74
HS	22	37	Th	248	75
$2.HS$	56	37	Fi_{23}	253	82
J_3	18	37	$Co1$	276	82
M_{24}	22	39	$2.Co1$	24	82
McL	21	42	J_4	1333	87
He	51	45	Fi'_{24}	781	106
Ru	378	50	B	4371	144
$2.Ru$	28	50	$2.B$	96256	144
M	196882	229			

Proposition 5.27. *Assume M is almost simple of type S, where S is a simple sporadic group. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1-3.*

Proof. Assume equation (3.1) holds for some $r \in \{s,t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$. Then by (5.3), $|Aut(S)| \geqslant 3^{m-1}$. It follows that $\Re_3(S) \leqslant 2m+1 \leqslant 2log_3(|Aut(S)|)+3 \leqslant g_3(S)$. Recall that V is an absolutely irreducible $\mathbb{F}_3\widehat{S}$-module with $dimV = 2m+1$ and the Frobenius-Schur indicator of V is $+$. By [14] and Lemma 5.26, we need to consider the following cases: $(S, dimV) = (M_{22}, 21)$, $(McL, 21)$, and $(Co_2, 23)$. If $(S, dimV) = (M_{22}, 21)$, then $M_{22} < A_{22} < \Omega_{21}(3)$, hence $N_G(S)$ is not maximal in G. If $(S, dimV) = (McL, 21)$, then S has two orbits with representatives $\langle x\rangle, \langle y\rangle$ and stabilizers $S_{\langle x\rangle} = L_3(4) : 2_2$ and $S_{\langle y\rangle} = M_{11}$ which are in different G-orbits. We also have $c_x = 12194, d_x = 10080$ and $c_y = 72809, d_y = 40590$, and we can check that (3.1) cannot hold in any of these cases. For $S : 2$, we also get the same result. Finally if $(S, dimV) = (Co_2, 23)$, then there exist two non-singular points in different G-orbits with representative $\langle x\rangle, \langle y\rangle$ with $S_{\langle x\rangle} = 2^{10} : M_{22} : 2$ and $S_{\langle y\rangle} = HS : 2$ with sizes $|\langle x\rangle S| = 46575 < 3^{m-1} = 3^{10}, |\langle y\rangle S| = 476928$. The parameters for $\langle y\rangle S$ are $c_y = 296450, d_y = 180477$. We can check that (3.1) cannot hold in this case. □

5.3.5. *Computation with GAP.* In this section we briefly demonstrate how to compute the parameters c and d defined in Section 3 by using GAP [10]. We still assume the hypotheses and notation from Section 4. Recall from section 5.1 that if $\langle x\rangle \in \mathfrak{E}_\xi(V)$ then

$$\Delta(x) = \mathfrak{E}_\xi(V) \cap x^\perp \text{ and } \Gamma(x) = \mathfrak{E}_\xi(V) \cap (V - x^\perp - \{\langle x\rangle\}).$$

Let M be a subgroup of G, where G is almost simple with socle $L = \Omega_{2m+1}(3), m \geqslant 3$. Note that $\mathfrak{E}_\xi(V), \xi = \pm$, is an L-orbit of non-singular points of type ξ. The parameters c and d are defined as follows:

$$d = |\langle x\rangle M \cap \Delta(x)| \text{ and } c = |\langle x\rangle M \cap \Gamma(x)| = |\langle x\rangle M| - d - 1.$$

As an example, let $M \cong S_8$, a symmetric group of degree 8 and $L = \Omega_{13}(3)$. Then M embeds into L via the Specht module V corresponding to the partition $(6, 2)$ (see Proposition 5.19). Let U be the permutation module for S_8 in characteristic 3. We know that V is a composition factor of the tensor product $U \otimes U$. Using MeatAxe to decompose $U \otimes U$ to obtain V. From this module, we can obtain the embedding of M into $\Omega(V)$ and also the quadratic form of V. Using this information, it is very easy to compute c and d.

```
>U:=PermutationGModule(SymmetricGroup(8),GF(3));;
♯ GF(3) is the field of size 3.
>T:=TensorProductGModule(U,U);
♯ The tensor product U ⊗ U.
>decomp:=MTX.CompositionFactors(T);
>V:=decomp[Position(List(decomp,x->MTX.Dimension(x)),13)];
♯ find the module of dimension 13.
>f:=MTX.InvariantBilinearForm(V);
♯ The bilinear form on V.
>gens:=MTX.Generators(V);
♯ The generators for M when embedded in Ω(V).
>M:=Group(gens);;
>ob:=OrbitsDomain(M,GF(3)^(MTX.Dimension(V)));;
♯ all orbits of M on V.
>leng:=List(ob,x->Size(x));;
♯ The lengths of all orbits of M on V;
>Positions(leng, 315);
♯ find the positions of orbits of length 315.
♯ for example, the positions are [54, 65, 72, 73]. These positions are not fixed.
> x := ob[54][1];; x * f * x;
♯ x is a representative for the orbit ob[54].
♯ x * f * x; is the norm of x.
♯ The different norms mean the points are in different L-orbits.
> op:=Orbit(G,GF(3)^(MTX.Dimension(V)),x);;
♯ op is the M-orbit ⟨x⟩M.
> d:=0;;for i in [1..Size(op)] do if op[i] * f * x = 0 * Z(3) then d := d + 1; fi;od;
♯ Z(3) is a generator for the field GF(3).
> c:=Size(op)-1-d;
```

For matrix groups, given the matrix generators 'gens', which can be obtained from the package 'atlasrep' or from the Atlas website [6]. The module is then constructed by using the GAP command 'V:=GModuleByMats(gens,GF(3))'.

```
> LoadPackage('atlasrep');
> DisplayAtlasInfo('G2(4)');
> gens:=AtlasGenerators('G2(4)',9).generators;;
♯ This is the 64-dimensional representation of G2(4) in characteristic 3.
> V:=GModuleByMats(gens);;
> M:=Group(gens);
```

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