# GROUPS WITH SOME ARITHMETIC CONDITIONS ON REAL CLASS SIZES

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ABSTRACT. Let $G$ be a finite group. An element $x \in G$ is a real element if $x$ and $x^{-1}$ are conjugate in $G$. For $x \in G$, the conjugacy class $x^G$ is said to be a real conjugacy class if every element of $x^G$ is real. In this paper, we show that if 4 divides no real conjugacy class sizes of a finite group $G$, then $G$ is solvable. We also study the structure of such groups in detail. This generalizes several results in the literature.

## 1. INTRODUCTION

It is well known that the arithmetic conditions on the conjugacy class sizes have strong influence on the structure of finite groups. For a full account, we refer the readers to an excellent survey by Camina and Camina [2]. In particular, the problem of recognizing the solvability of finite groups using arithmetic conditions on class sizes has attracted many authors. A classical result in finite group theory says that 'For a fixed prime $p$, if $p$ does not divide any conjugacy class sizes of a finite group $G$, then the Sylow $p$-subgroup of $G$ is an abelian direct factor of $G$.' (See [5, Proposition 4] for instance). Especially, if $p = 2$, then the group $G$ is solvable. This result has been generalized by many authors in the literature. For example, A.R. Camina [1] showed that the Sylow $p$-subgroup of $G$ is a direct factor if the conjugacy class size of every $p'$-element of $G$ is prime to $p$. Recall that an element $x$ in a finite group $G$ is **real** if $x$ and $x^{-1}$ are conjugate in $G$. Equivalently, $x \in G$ is real if and only if $\chi(x)$ is real, that is, $\chi(x) \in \mathbb{R}$, for any $\chi \in \mathrm{Irr}(G)$, where $\mathrm{Irr}(G)$ denotes the set of all complex irreducible characters of $G$. Of course, if $x \in G$ is real then every element in the conjugacy class $K := x^G$ containing $x$ is also real and we say that $K$ is a **real conjugacy class** and $|K|$ is a **real class size**. It turns out that the aforementioned results hold by restricting to real class sizes. For example, in [6, Theorem 6.1], Dolfi, Navarro and Tiep showed that the Sylow 2-subgroup of a finite group $G$ is normal in $G$ if all real conjugacy classes of $G$ have odd size. (For odd primes, a similar result has been obtained by Guralnick, Navarro and Tiep [10] but there is some complication for the prime $p = 3$.) In [12], Navarro, Sanus and Tiep studied the structure of finite groups whose all real class sizes are 2-powers. In particular, all these groups are solvable. In this paper, we generalize a result of Chillag and Herzog [5, Proposition 5] by proving the following.

**Theorem A.** *Let $G$ be a finite group. If the conjugacy class size of every odd prime power order real element in $G$ is a $2$-power or not divisible by $4$, then $G$ is solvable.*

This also gives a generalization to [7, Theorem 3.1]. We note that the analogous problem for complex irreducible characters does not hold as 4 divides no irreducible complex character degrees of the alternating group of degree 7, but this group is clearly not solvable.

We now study in detail the structure of finite groups whose conjugacy class sizes of odd prime power order real elements are not divisible by 4. Note that $\mathbf{O}^{2'}(G)$ is the smallest normal subgroup of $G$ such that the quotient $G/\mathbf{O}^{2'}(G)$ is a group of odd order.

**Theorem B.** *Let $G$ be a finite group. If the conjugacy class size of every odd prime power order real element in $G$ is not divisible by $4$, then $\mathbf{O}^{2'}(G)$ is $2$-nilpotent and $G/\mathbf{O}_{2'}(G)$ is $2$-closed.*

If $n$ is a positive integer and $p$ is a prime, then we can write $n = n_p n_{p'}$, where $n_p$ is a power of $p$ and $n_{p'}$ is relative prime to $p$. We will call $n_p$ the $p$-part of $n$. One consequence of Theorem B is that if the 2-part of the conjugacy class sizes of all noncentral real elements of $G$ is exactly 2, then $\mathbf{O}^{2'}(G)$ is 2-nilpotent. To generalize this result, we dare to make the following conjecture.

**Conjecture C.** *Suppose that the sizes of all noncentral real conjugacy classes of a finite group $G$ have the same $2$-part. Then $\mathbf{O}^{2'}(G)$ is $2$-nilpotent. In particular, $G$ is solvable.*

The group $\mathrm{PSL}_3(2)$ shows that we cannot restrict the hypothesis of Conjecture C to noncentral odd order real conjugacy classes since this group has only one noncentral odd order real conjugacy class with size $7 \cdot 2^3$. Recently, it has been proved in [3] that if the sizes of noncentral conjugacy classes of a group G have the same non-trivial $p$-part for some prime $p$, then $G$ is solvable and has a normal $p$-complement. This interesting result which gives a weak analogue to the famous J. Thompson's Theorem on character degrees raises our hope of the veracity of our conjecture. One application of Conjecture C, if true, is a positive answer to a question due to G. Navarro (see [13, Question]) which asked whether a group $G$ is solvable if $|\mathbf{C}_G(x)| = |\mathbf{C}_G(y)|$ for any noncentral real elements $x$ and $y$ in $G$. In other words, whether a group $G$ is solvable if it has at most two real class sizes. As suspected, we will make use of the classification of finite simple groups for the proof of Theorem A. It would be interesting if one can find a classification free proof of this result.

**Notation.** By a group, we mean a finite group. Let $G$ be a group. If $g \in G$, then we write $o(g)$ for the order of $g$. The greatest common divisor of two integers $a$ and $b$ is $\gcd(a, b)$. The $m^{th}$ cyclotomic polynomial in variable $q$ is denoted by $\Phi_m(q)$. We write $\mathrm{Z}(G)$ for the center of $G$. The symmetric and alternating groups of degree $n$ are denoted by $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$, respectively. Finally, we denote by $\mathbb{Z}_m$ a cyclic group of order $m$. Unexplained notation is standard.

## 2. Preliminaries

We collect here some basic properties of conjugacy classes in a finite group. The following results are well known and easy to verify.

**Lemma 2.1.** *Let $N$ be a normal subgroup of a group $G$. Then*

    *(1) If $x \in N$, then $|x^N|$ divides $|x^G|$.*
    *(2) If $x \in G$, then $|(Nx)^{G/N}|$ divides $|x^G|$.*

The next result is elementary. For completeness we present a proof.

**Lemma 2.2.** *Let $N \leq \mathrm{Z}(G)$ be a central subgroup of a finite group $G$. If $x \in G$ such that $(o(x), |N|) = 1$, then $|x^G| = |(Nx)^{G/N}|$.*

*Proof.* Assume that $x \in G$ with $(o(x), |N|) = 1$. Let $H$ be the full inverse image in $G$ of $\mathbf{C}_{G/N}(Nx)$. Then $N \leq \mathbf{C}_G(x) \leq H \leq G$ and therefore $|x^G| = |G : \mathbf{C}_G(x)|$ and $|(Nx)^{G/N}| = |G/N : H/N| = |G : H|$. Hence we need to show that $H = \mathbf{C}_G(x)$. As $\mathbf{C}_G(x) \leq H$, it suffices to show that $H \leq \mathbf{C}_G(x)$. Let $g \in H$. We have that $Nx^g = Nx$ and hence $x^g = xz$ for some $z \in N$. Since $x^g$ and $x$ have the same order, say $n$, we deduce that $(x^g)^n = x^n z^n$, so $z^n = 1$. Therefore, $o(z)$ divides $\gcd(o(x), |N|) = 1$, so $z = 1$. Hence $x^g = x$ which means that $g \in \mathbf{C}_G(x)$ and so $H \leq \mathbf{C}_G(x)$ as required. $\qquad\square$

We now turn our attention to real elements and real class sizes.

**Lemma 2.3.** ([7, Lemma 2.4]). *Let $G$ be a finite group.*

    *(1) If $x \in G$ is real and $|x^G|$ is odd, then $x^2 = 1$.*
    *(2) If $x \in G$ is real, then every power of $x$ is a real element of $G$.*
    *(3) The identity is the unique real element of $G$ if and only if $|G|$ is odd.*
    *(4) If $N \trianglelefteq G$ and $|G/N|$ is odd, then real elements of $G$ are real elements of $N$.*

**Lemma 2.4.** *If $x, g \in G$ with $x^g = x^{-1}$, then $x^t = x^{-1}$ for a 2-element $t$ in $\langle g \rangle$.*

*Proof.* If $o(g) = 2^k m$, where $m$ is odd, then the 2-element $t := g^m$ will work. $\quad\square$

The following lemma gives some conditions to guarantee that a real element of the quotient group $G/N$ can be lifted to a real element of the whole group $G$.

**Lemma 2.5.** *Suppose that $N \trianglelefteq G$ and that $Nx$ is a real element in $G/N$.*

    *(1) If $(o(x), |N|) = 1$, then $x$ is real in $G$.*
    *(2) If the order of $Nx$ in $G/N$ is prime to $|N|$, then $Nx = Ny$ for some real element $y$ of $G$ (of odd order if the order of $Nx$ is odd).*
    *(3) If $|N|$ or the order of $Nx$ in $G/N$ is odd, then $Nx = Ny$ for some real element $y$ of $G$ (of odd order if the order of $Nx$ is odd).*

*Proof.* These statements can be found in [12, Lemma 3.1] and [10, Lemma 2.2]. $\quad\square$

As we are dealing with odd prime power order real elements, we will need the following result which is an easy consequence of Lemma 2.5(3).

**Lemma 2.6.** *Let $N$ be a normal subgroup of a group $G$ and let $p$ be an odd prime. If $Nx$ is a $p$-power order real element in $G/N$, then $Nx = Ny$ for some $p$-power order real element $y$ in $G$.*

*Proof.* Assume that $Nx \in G/N$ is a real element of order $p^a$ with $a \geq 0$. If $a = 0$, then the result is clear. So, we assume that $a \geq 1$. As $p$ is odd, by Lemma 2.5(3) we obtain that $Nx = Nz$ for some real element $z$ in $G$ of odd order. It follows that $z^{p^a} \in N$ since $Nx = Nz$ has order $p^a$ in $G/N$. Write $o(z) = p^b k$, where $p \nmid k$. Since $\gcd(p^a, k) = 1$, we deduce that $p^a u + k v = 1$ for some integers $u$ and $v$. Now we

have that $Nx = Nz = Nz^{kv}$ as $z^{p^a u} \in N$. By Lemma 2.3(2), we know that $z^{kv}$ is real in $G$. Also $(z^{kv})^{p^b} = 1$ since $o(z) = p^b k$. Thus $y = z^{vk}$ is a $p$-power order real element with $Nx = Ny$ as wanted. $\qquad\square$

The next result describes the structure of groups with no nontrivial real elements of odd order.

**Lemma 2.7.** ([6, Proposition 6.4]). *Let $G$ be a finite group. Then every nontrivial real element in $G$ has even order if and only if $G$ has a normal Sylow 2-subgroup.*

We also need the following well known result due to Zsigmondy.

**Lemma 2.8.** (Zsigmondy Theorem [15]). *Let $q$ and $n$ be integers with $q \geq 2$, $n \geq 3$. Assume that $(q, n) \neq (2, 6)$. Then $q^n - 1$ has a prime divisor $\ell$ such that $\ell$ does not divide $q^m - 1$ for $m < n$. Moreover $\ell \equiv 1 \pmod{n}$ and if $\ell \mid q^k - 1$, then $n \mid k$.*

Such an $\ell$ is called a *primitive prime divisor*. We denote by $\ell_n(q)$ the smallest primitive prime divisor of $q^n - 1$ for fixed $q$ and $n$.

## 3. REAL ELEMENTS IN SIMPLE GROUPS

Let $G$ be a group. We say that an element $x \in G$ is *good* if it is a real element of odd prime power order whose class size is divisible by 4 and the order of $x$ is prime to $|Z(G)|$. Notice that if the center of $G$ is trivial, then the latter condition in the definition above is trivially satisfied. In this section, we prove the following result.

**Proposition 3.1.** *Every nonabelian simple group $S$ possesses a good element.*

*Proof.* Using the classification of simple groups, we consider the following cases.
**Case 1:** $S \cong \mathrm{Alt}(n)$, an alternating group with degree $n \geq 5$. Let $x = (1, 2, 3, 4, 5)$ and $g = (2, 5)(3, 4)$ be two elements in $S$. We can see that $o(x) = 5$ and $x^g = x^{-1}$, so $x$ is a real element of order 5. We have that $\mathbf{C}_S(x) \cong \mathbb{Z}_5$, if $5 \leq n \leq 6$; and $\mathbf{C}_S(x) \cong \mathbb{Z}_5 \times \mathrm{Alt}(n - 5)$, if $n \geq 7$. Thus

$$|x^S| = \begin{cases} \frac{1}{5}n(n-1)(n-2)(n-3)(n-4), & \text{if } n \geq 7; \\ \frac{1}{10}n(n-1)(n-2)(n-3)(n-4), & \text{if } n \in \{5, 6\}. \end{cases}$$

It follows that $4 \mid |x^S|$ in any case, so $x$ is a good element in $S$.
**Case 2:** $S$ is a sporadic simple group or the Tits group. These cases can be checked directly by using [4].

We now consider the following set up. Let $S$ be a simple group of Lie type in characteristic $p$ defined over a field of size $q = p^f$ with $S \not\cong {}^2\mathrm{F}_4(2)'$. Let $\mathcal{G}$ be a simple, simply connected algebraic group in characteristic $p$ and let $F : \mathcal{G} \to \mathcal{G}$ be a Frobenius map such that if $G := \mathcal{G}^F$, then $S \cong G/Z(G)$. It follows from [14, Proposition 3.1] that every semisimple element of $G$ is real unless $\mathcal{G}$ is of type $\mathrm{A}_n$ with $n > 1$, $\mathrm{D}_{2n+1}$ or $\mathrm{E}_6$. We will use the bar convention in $G/Z(G)$. By Lemma 2.2, if $x$ is a good element in $G$, then $\bar{x}$ is a good element in $S$. Thus we need to find an odd prime power order real element $x \in G$ such that $o(x)$ is prime to $|Z(G)|$ and 4 divides $|x^G|$. For $\epsilon = \pm$, we use the convention that $\mathrm{SL}_n^\epsilon(q)$ is $\mathrm{SL}_n(q)$ if $\epsilon = +$ and $\mathrm{SU}_n(q)$ if $\epsilon = -$; similar convention applies to $\mathrm{GL}_n^\epsilon(q)$. We also write $\mathrm{E}_6{}^+(q)$ for $\mathrm{E}_6(q)$ and $\mathrm{E}_6^-(q)$ for ${}^2\mathrm{E}_6(q)$.
**Case 3:** Assume that $\mathcal{G}$ is of type $\mathrm{A}_{n-1}$ with $n \geq 2$. Then $S \cong \mathrm{PSL}_n^\epsilon(q)$ and $G \cong \mathrm{SL}_n^\epsilon(q)$ with $n \geq 2$, $q = p^f$ and $\epsilon = \pm$.

(1) Assume that $S \cong \mathrm{PSL}_2(q)$, where $q \geq 5$ is odd. As $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathrm{Alt}(5)$, we can assume that $q \geq 7$. We know that $S$ has dihedral subgroups of order $q \pm 1$ with cyclic subgroups of order $(q \pm 1)/2$. Firstly, assume that $q \equiv 1 \pmod 4$. Then $(q+1)/2$ is odd and hence it has an odd prime divisor $r$. It follows that $S$ has a real element $x$ of order $r$ with $|\mathbf{C}_S(x)| = (q+1)/2$. Hence $|x^S| = q(q-1)$ which is divisible by 4 since $q \equiv 1 \pmod 4$. Finally, assume that $q \equiv 3 \pmod 4$. In this case $(q-1)/2$ is odd and $q(q+1)$ is divisible by 4. Now let $x$ be a real element of odd prime order $r \mid (q-1)/2$ and apply the argument above, we deduce that $x$ is good.

(2) Assume that $n \geq 3$ and $q$ is odd. It follows from the proof of [10, Lemma 4.4] that $G \cong \mathrm{SL}_n^\epsilon(q)$ has a real element $x$ of $p$-power order with class size

$$|x^G| = q^{n(n-3)/2}(q + \epsilon 1) \prod_{i=3}^{n} (q^n - \epsilon^n 1).$$

Since $|\mathrm{Z}(G)| = \gcd(n, q - \epsilon 1)$ is prime to $p$, it suffices to show that $|x^G|$ is divisible by 4. We now have that $|x^G|$ is divisible by $(q + \epsilon 1)(q - \epsilon 1) = q^2 - 1$. Since $q$ is odd, we deduce that $q^2 - 1$ is divisible by 4 so is $|x^G|$. Hence $x$ is good in $G$.

(3) Assume $S \cong \mathrm{PSL}_n(q)$ with $n \geq 2, q = 2^f$. Embed $\mathrm{SL}_2(q) \times \mathrm{SL}_{n-2}(q)$ in $G = \mathrm{SL}_n(q)$. Let $B \in \mathrm{SL}_2(q)$ such that $B$ is of prime power order and $o(B) \mid q + 1$. Define

$$x = \begin{pmatrix} B & 0 \\ 0 & I_{n-2} \end{pmatrix},$$

with $I_{n-2} \in \mathrm{SL}_{n-2}(q)$ being the identity matrix. Since $B$ is a semisimple real element of $\mathrm{SL}_2(q)$, $x$ is a semisimple real element of $\mathrm{SL}_n(q)$. Now for $n = 2$ we have $q \geq 4$, so $|x^G| = q(q-1)$ is divisible by 4. Finally, for $n \geq 3$ we have $|x^G| = t|B^{\mathrm{SL}_2(q)}|$, where $t$ denotes the number of decompositions of an $n$-dimensional vector space over $\mathbb{F}_q$ into the direct sum of a 2-dimensional and an $n-2$-dimensional subspaces. Easy calculation shows that

$$t = \frac{|\mathrm{GL}_n(q)|}{|\mathrm{GL}_2(q)| \cdot |\mathrm{GL}_{n-2}(q)|} = q^{2(n-2)} \frac{(q^n - 1)(q^{n-1} - 1)}{(q-1)(q^2 - 1)},$$

so $4 \mid t$ and thus $4 \mid |x^G|$ as required. Furthermore, $\gcd(o(x), |\mathrm{Z}(G)|) = 1$, therefore $x$ is a good element in $G$.

(4) Assume $G \cong \mathrm{SU}_n(q)$ with $n \geq 3$ and $q = 2^f$. Assume first that $n = 3$. Since $\mathrm{PSU}_3(2)$ is not simple, we can assume that $q \geq 4$. Let $B \in \mathrm{SU}_2(q) = \mathrm{SL}_2(q)$ such that $B$ is of prime power order and $o(B) \mid q - 1$. Embed $\mathrm{SU}_2(q)$ in $G$ and let

$$x = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $x$ is a semisimple real element of $G$. We have that $|x^G| = t|B^{\mathrm{SU}_2(q)}|$, where $t$ denotes the number of decompositions of a 3-dimensional non-degenerate unitary space over $\mathbb{F}_{q^2}$ into the orthogonal direct sum of a 2-dimensional and a 1-dimensional subspaces and

$$t = \frac{|\mathrm{GU}_3(q)|}{|\mathrm{GU}_2(q)| \cdot |\mathrm{GU}_1(q)|} = q^2 \frac{(q^3 + 1)(q^2 - 1)}{(q+1)(q^2 - 1)},$$

so $4 \mid t$ and thus $4 \mid |x^G|$ as required. Since $o(x) = o(B) \mid q - 1$ and $|\mathrm{Z}(G)| = \gcd(3, q + 1)$, we see that $\gcd(o(x), |\mathrm{Z}(G)|) = 1$, so $x$ is a good element in $G$.

Assume next that $n \geq 4$. Let $m = 2\lfloor n/2 \rfloor \geq 4$. Assume first that $(n,q) \neq (6,2), (7,2)$. Then $\ell_{mf}(2) > 2$ exists. It follows from the proof of Lemma 3.5 in [6] that there exists a real element $x$ of order $\ell_{mf}(2)$ with $x \in Sp_m(q) \leq G$ and

$$|\mathbf{C}_G(x)| = \begin{cases} (q^n - 1)/(q+1), & \text{if } n \equiv 0 \pmod 4; \\ (q^{n/2} + 1)^2/(q+1), & \text{if } n \equiv 2 \pmod 4; \\ (q^{n-1} - 1), & \text{if } n \equiv 1 \pmod 4; \\ (q^{(n-1)/2} + 1)^2, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Hence $|x^G|$ is divisible by $q^{n(n-1)/2}$ and so by 4. Moreover $|\mathbf{Z}(G)| = \gcd(n, q+1)$ dividing $q^2 - 1 = 2^{2f} - 1$ and thus $\ell_{mf}(2)$ cannot divide $|\mathbf{Z}(G)|$ as $m \geq 4$. Therefore, $x$ is good in $G$. Assume next that $(n,q) = (6,2)$. Using [8], $S$ has a real element $x$ of order 7 with $|\mathbf{C}_S(x)| = 7$. Obviously 4 divides $|x^S|$ and thus $x$ is good in $S$. Assume now that $(n,q) = (7,2)$. We can embed $\mathrm{PSU}_6(2)$ in $\mathrm{PSU}_7(2)$ and we can choose $x$ to be a real element of order 7 as in the previous case. In this case, we have $|\mathbf{C}_S(x)| = 63$ and so 4 divides $|x^S|$. Hence $x$ is good in $S$.

**Case 4:** Assume $\mathcal{G}$ is of type $\mathrm{B}_n$ or $\mathrm{C}_n$ with $n \geq 2$. Then $G \cong \mathrm{Spin}_{2n+1}(q)$ or $\mathrm{Sp}_{2n}(q)$. In both cases, we have $|\mathbf{Z}(G)| = \gcd(2, q-1)$. Assume that $(n,q) \neq (3,2)$. By [11, Lemma 2.4], $G$ has a semisimple element $x$ of order $\ell_{2nf}(p)$ with $|\mathbf{C}_G(x)| = q^n + 1$. As $\ell_{2n}(q) > 2$ we deduce that $o(x)$ is prime to $|\mathbf{Z}(G)|$. We need to verify that $|x^G|$ is divisible by 4. If $q$ is even then $|x^G|$ is divisible by $q^{n^2}$ and so by 4 as $n \geq 2$. Now assume that $q$ is odd. In this case, $|x^G|$ is always divisible by $q^2 - 1$. Since $q$ is odd, $q^2 - 1$ is divisible by 4, so $|x^G|$ is divisible by 4. Now assume $(n,q) = (3,2)$. Then $G \cong S \cong \mathrm{Sp}_6(2)$. Using [4], we can check that $G$ has a semisimple element of order 7 whose class size is divisible by 4.

**Case 5:** Assume that $\mathcal{G}$ is of type $\mathrm{D}_n$ with $n \geq 4$. Then $G \cong \mathrm{Spin}_{2n}^\epsilon(q)$ with $n \geq 4$ and $\epsilon = \pm$. Assume first that $n$ is even and $(n,q) \neq (4,2)$. By [11, Lemma 2.4], $G$ has a semisimple element $x$ of order $\ell_{2(n-1)f}(p)$ with $|\mathbf{C}_G(x)| = (q^{n-1} + 1)(q + \epsilon 1)$. As $n \geq 4$, it is easy to check that $|x^G|$ is divisible by 4. Moreover, as $|\mathbf{Z}(G)| = \gcd(4, q^n - \epsilon 1)$ and $o(x) = \ell_{2(n-1)f}(p)$ is odd, we deduce that the order of $x$ is prime to $|\mathbf{Z}(G)|$. Now assume that $(n,q) = (4,2)$. Then $S \cong \mathrm{P}\Omega_8^\epsilon(2)$ and the results follows easily by checking [4].

Assume that $n \geq 5$ is odd. Embed $H = \mathrm{Spin}_{2n-2}^-(q)$ in $G$. By [11, Lemma 2.4], there exists a semisimple element $x \in H$ of order $\ell_{(2n-2)f}(p)$ with $|\mathbf{C}_H(x)| = q^{n-1} + 1$. Note that $x$ is real in $H$ as $n - 1$ is even. So, $x$ is real in $G$. As in the previous case, the order of $x$ is prime to $|\mathbf{Z}(G)|$. Furthermore, we have that $|\mathbf{C}_G(x)| = (q^{n-1} + 1)(q + \epsilon 1)$. It is now easy to check that $|x^G|$ is divisible by 4.

**Case 6:** Assume $\mathcal{G}$ is of exceptional type.

(1) Assume that $G \cong {}^2\mathrm{G}_2(q)$ with $q = 3^{2m+1}, m \geq 1$. Then $|\mathbf{Z}(G)| = 1$. By [11, Lemma 2.3], $G$ has a semisimple element $x$ of order $\ell_{2m+1}(3)$ with $|\mathbf{C}_G(x)| = q - 1$. Now $|x^G| = q^3(q^3 + 1)$ is divisible by $q + 1 = 3^{2m+1} + 1$. Since $3^{2m+1} + 1$ is divisible by 4, we deduce that $|x^G|$ is divisible by 4 as required.

Similar argument applies to the group ${}^2\mathrm{B}_2(q)$ and ${}^2\mathrm{F}_4(q)$, where $q = 2^{2m+1}$ and $m \geq 1$, with semisimple elements of order $\ell_{2m+1}(2)$ and $\ell_{6(2m+1)}(2)$, respectively.

(2) Assume $\mathcal{G}$ is of type $\mathrm{E}_6$. We have that $G \cong \mathrm{E}_6^\epsilon(q)_{sc}$ with $\epsilon = \pm$ and $|\mathbf{Z}(G)| = (3, q - \epsilon 1)$. Embed $H = \mathrm{F}_4(q)$ in $G$. By [11, Lemma 2.3], $H$ has a semisimple and hence real element of order $\ell_{12f}(p)$ with $|\mathbf{C}_H(x)| = \Phi_{12}(q)$. As $|\mathbf{Z}(G)| \leq 3$ and $\ell_{12f}(p) \geq 13$ by Lemma 2.8, we see that $o(x)$ is prime to $|\mathbf{Z}(G)|$. Now it

follows from the proof of [11, Lemma 2.3] that $|\mathbf{C}_G(x)| = \Phi_{12}(q)(q^2 + \epsilon q + 1)$. Since $\Phi_{12}(q)(q^2 + \epsilon q + 1) = (q^4 - q^2 + 1)(q^2 + \epsilon q + 1)$ is always odd and $|\mathrm{E}_6^\epsilon(q)|$ is divisible by 4, we obtain that $|x^G|$ is divisible by 4.

(3) Assume $\mathcal{G}$ is of type $\mathrm{E}_7$. Then $|\mathbf{Z}(G)| = (2, q - 1)$. By [11, Lemma 2.3], $G$ has a semisimple element $x$ of order $\ell_{mf}(p)$ with $|\mathbf{C}_G(x)| = \Phi_{18}(q)\Phi_2(q)$, where $m = 18$. Observe first that $o(x)$ is prime to $|\mathbf{Z}(G)|$. Now if $q$ is even, then $|x^G|$ is divisible by $q^{63}$ and hence by 4. Otherwise, if $q$ is odd, then $|x^G|$ is divisible by $q^2 - 1$. As $q$ is odd, we deduce that $q^2 - 1$ is divisible by 4.

The same argument applies to the groups $\mathrm{E}_8(q), \mathrm{F}_4(q), {}^3\mathrm{D}_4(q)$ and $\mathrm{G}_2(q)(q > 2)$, where $m = 30, 12, 12$ and $6$, respectively. The proof is now complete. $\square$

## 4. Arithmetic conditions on real class sizes

We say that a group $G$ has property $\mathcal{T}$ if the conjugacy class size of every odd prime power order real element in $G$ is not divisible by 4, i.e., if $x$ is a real element of order $p^a$ for some odd prime $p$, then $4 \nmid |x^G|$. Similarly, a group $G$ has property $\tilde{\mathcal{T}}$ if the conjugacy class size of every odd prime power order real element in $G$ is either a 2-power or not divisible by 4. It follows from the definitions that if $G$ has property $\mathcal{T}$, then it has property $\tilde{\mathcal{T}}$.

**Lemma 4.1.** *Suppose that $N \trianglelefteq G$ and that $G$ has property $\mathcal{T}$ or $\tilde{\mathcal{T}}$. Then both $N$ and $G/N$ have property $\mathcal{T}$ or $\tilde{\mathcal{T}}$, respectively.*

*Proof.* Assume that $N \trianglelefteq G$ and $x \in N$. Suppose that $x$ is an odd prime power order real element in $N$. Then clearly $x$ is an odd prime power order real element in $G$. Since $|x^N|$ divides $|x^G|$ by Lemma 2.1(1), we deduce that if $4 \nmid |x^G|$, then $4 \nmid |x^N|$; and if $|x^G|$ is a 2-power, then so is $|x^N|$. This shows that if $G$ has property $\mathcal{T}$ or $\tilde{\mathcal{T}}$, then $N$ also satisfies the same property. Assume now that $Nx$ is an odd prime power order real element in $G/N$. By Lemma 2.6, $Nx = Ny$ for some odd prime power order real element $y \in G$. By Lemma 2.1(2) we obtain that

$$|(Nx)^{G/N}| = |(Ny)^{G/N}|$$

divides $|y^G|$. Clearly, if $|y^G|$ is a 2-power or $4 \nmid |y^G|$, then $|(Nx)^{G/N}|$ is also a 2-power or $4 \nmid |(Nx)^{G/N}|$, respectively. Therefore, we conclude that if $G$ has property $\mathcal{T}$ or $\tilde{\mathcal{T}}$, then $G/N$ has property $\mathcal{T}$ or $\tilde{\mathcal{T}}$, respectively. $\square$

Now we prove Theorem A which we restate here.

**Theorem 4.2.** *If $G$ has property $\tilde{\mathcal{T}}$, then $G$ is solvable.*

*Proof.* We prove this theorem by induction on the order of $G$. Assume first that $G$ has a nontrivial normal subgroup $N$ such that $N \neq G$. By Lemma 4.1, both $N$ and $G/N$ are solvable by induction hypotheses. Hence $G$ is solvable. Thus we can assume that $G$ is nonabelian simple. By [9, Lemma 4.3.2], $\{1\}$ is the only conjugacy class of $G$ which has prime power size. Therefore, 4 does not divide the conjugacy class size of any odd prime power order real element in $G$. Now Proposition 3.1 yields a contradiction. This contradiction shows that $G$ is solvable. $\square$

Here is a direct consequence of the theorem above.

**Corollary 4.3.** *If $G$ has property $\mathcal{T}$, then $G$ is solvable.*

Let $p$ be a prime. Recall that a group $G$ is said to be $p$-closed if some Sylow $p$-subgroup of $G$ is normal in $G$; and $G$ is said to be $p$-nilpotent if it has a normal $p$-complement. We denote by $\mathbf{O}^{p'}(G)$ the smallest normal subgroup of $G$ such that the quotient $G/\mathbf{O}^{p'}(G)$ is a $p'$-group, that is, its order is prime to $p$. Also $\mathbf{O}_p(G)$ is the largest normal $p$-subgroup of $G$. Furthermore, if $G$ is solvable and $\mathbf{O}_{p'}(G) = 1$, then $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$.

**Proposition 4.4.** *Let $G$ be a group. Suppose that $G$ has property $\mathcal{T}$ and that $G = \mathbf{O}^{2'}(G)$. Then $G$ is 2-nilpotent.*

*Proof.* Assume that $G = \mathbf{O}^{2'}(G)$ and that the conjugacy class size of every odd prime power order real element of $G$ is not divisible by 4. If $G$ is a 2-group, then the conclusion is trivially true. Hence we may assume that $G$ is not a 2-group. Clearly the Sylow 2-subgroup of $G$ is not normal in $G$ since $G$ has no nontrivial factor group of odd order. By Corollary 4.3, we know that $G$ is solvable. We will show that $G$ has a normal 2-complement by induction on $|G|$. We consider the following cases.

(1) Assume first that $\mathbf{O}_{2'}(G) = 1$. Let $U := \mathbf{O}_2(G) \trianglelefteq G$. As $G$ is solvable, by [9, Theorem 6.3.2] we have that $\mathbf{C}_G(U) \leq U$. In particular, $U$ is nontrivial. As $G$ is not 2-closed, $G$ possesses some nontrivial odd order real element by Lemma 2.7. Applying Lemma 2.3(2), there is a real element $x \in G$ of odd prime order $p$. Let the generalized centralizer of $x$ in $G$ defined by

$$\mathbf{C}_G^*(x) := \{g \in G \mid x^g \in \{x, x^{-1}\}\} \leq G.$$

Then $\mathbf{C}_G(x) \trianglelefteq \mathbf{C}_G^*(x)$ and $|\mathbf{C}_G^*(x) : \mathbf{C}_G(x)| = 2$. Since

$$|x^G| = |G : \mathbf{C}_G^*(x)| \cdot |\mathbf{C}_G^*(x) : \mathbf{C}_G(x)| = 2|G : \mathbf{C}_G^*(x)|$$

and $4 \nmid |x^G|$ by our hypothesis, we obtain that $|G : \mathbf{C}_G^*(x)|$ is odd. It follows that $P \leq \mathbf{C}_G^*(x)$ for some Sylow 2-subgroup $P$ of $G$. As $U \trianglelefteq G$, we have that $U \leq P \leq \mathbf{C}_G^*(x)$. If $U \leq \mathbf{C}_G(x)$, then $x \in \mathbf{C}_G(U) \leq U$, a contradiction. Therefore, there exists an element $u \in U$ such that $x^u = x^{-1}$. It follows that $x^{-2} = x^{-1}x^u = [x, u] \in U$. As $\langle x \rangle \cap U = 1$, we deduce that $x^{-2} = 1$, which is impossible. This contradiction shows that $\mathbf{O}_{2'}(G)$ is nontrivial.

(2) Assume that $\mathbf{O}_{2'}(G) \neq 1$. Let $\bar{G} = G/\mathbf{O}_{2'}(G)$. Observe that $\mathbf{O}^{2'}(\bar{G}) = \bar{G}$. Also $\bar{G}$ has property $\mathcal{T}$ by Lemma 4.1. Thus $\bar{G}$ has a normal 2-complement by induction and so $G$ has a normal 2-complement as required. $\square$

We now prove Theorem B which we restate here for the reader's convenience.

**Theorem 4.5.** *Let $G$ be a finite group. If $G$ has property $\mathcal{T}$, then $\mathbf{O}^{2'}(G)$ is 2-nilpotent and $G/\mathbf{O}_{2'}(G)$ is 2-closed.*

*Proof.* Suppose that $G$ has property $\mathcal{T}$. We first observe that $H := \mathbf{O}^{2'}(G)$ is a characteristic subgroup of $G$ and it also has property $\mathcal{T}$ by Lemma 4.1. Since $\mathbf{O}^{2'}(H) = H$ has property $\mathcal{T}$, it follows from Proposition 4.4 that $H$ is 2-nilpotent, i.e., it has a normal 2-complement $K$. Since $K$ is characteristic in $H$ and $H$ is characteristic in $G$, we deduce that $K$ is characteristic in $G$. In particular, $K \trianglelefteq G$ and so $K \leq \mathbf{O}_{2'}(G)$. Let $P \in \mathrm{Syl}_2(H)$. As $G/H$ is of odd order, we have that $P \in \mathrm{Syl}_2(G)$ and $H = PK \trianglelefteq G$. By Frattini's argument, we have that

$$G = \mathbf{N}_G(P)K = \mathbf{N}_G(P)\mathbf{O}_{2'}(G).$$

Hence $G/\mathbf{O}_{2'}(G)$ has a normal Sylow 2-subgroup as wanted. $\square$

The following example shows that we cannot strengthen Proposition 4.4 and Theorem 4.5 by changing the condition that '$G$ has property $\mathcal{T}$' to the weaker condition '$G$ has property $\tilde{\mathcal{T}}$'.

**Example 4.6.** Let $G \cong \mathrm{Sym}(4)$ be the symmetric group of degree 4. We can see that $G = \mathbf{O}^{2'}(G), \mathbf{O}_{2'}(G) = 1$ and the conjugacy class sizes of odd prime power order real elements in $G$ are 1 and $2^3$. Thus $G$ has property $\tilde{\mathcal{T}}$ but not $\mathcal{T}$. Furthermore, $G$ is neither 2-closed nor 2-nilpotent.

## References

[1] A.R. Camina, Arithmetical conditions on the conjugacy class numbers of a finite group, *J. London Math. Soc.,* (2) **5** (1972), 127–132.

[2] A.R. Camina and R.D. Camina, The influence of conjugacy class sizes on the structure of finite groups: A survey, *Asian-European J. Math.,* **4** (2011), 559–588.

[3] C. Casolo, S. Dolfi and E. Jabara, Finite groups whose noncentral class sizes have the same p-part for some prime p, *Israel J. Math.,* **192** (2012), 197-219.

[4] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups,* Oxford University Press, (Eynsham, 1985).

[5] D. Chillag and M. Herzog, On the length of the conjugacy classes of finite groups, *J. Algebra,* **131** (1990), 110–125.

[6] S. Dolfi, G. Navarro and P.H. Tiep, Primes dividing the degrees of the real characters, *Math. Z.,* **259** (2008), 755–774.

[7] S. Dolfi, E. Pacifici and L. Sanus, Finite groups with real conjugacy classes of prime size, *Israel J. Math.,* **175** (2010), 179–189.

[8] The GAP Group, GAP - Groups, Algorithms, and Programming, Version 4.4.10, 2007. http://www.gap-system.org.

[9] D. Gorenstein, *Finite groups,* 2nd edition, Chelsea Publishing Company (New York, 1980).

[10] R. Guralnick, G. Navarro and P.H. Tiep, Real class sizes and real character degrees, *Math. Proc. Cambridge Philos. Soc.,* **150** (2011), 47–71.

[11] A. Moretó and P.H. Tiep, Prime divisors of character degrees, *J. Group Theory,* **11** (2008), 341–356.

[12] G. Navarro, L. Sanus and P.H. Tiep, Real characters and degrees, *Israel J. Math.,* **171** (2009), 157–173.

[13] L. Sanus, Real characters and degrees: an overview, in: *Ischia group theory 2008*, World Sci. Publ. (Hackensack, NJ, 2009), pp. 243-247.

[14] P.H. Tiep and A.E. Zalesski, Real conjugacy classes in algebraic groups and finite groups of Lie type, *J. Group Theory,* **8** (2005), no. 3, 291–315.

[15] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.,* **3** (1892), 265–284. (in German)

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