Chapter Goals
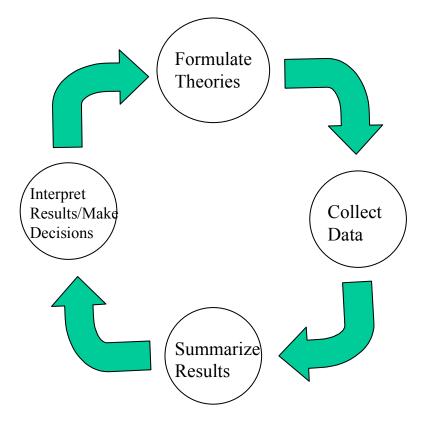
Here we will discuss methods to make decisions about a population mean.

Making Decisions About a Population Mean

1. State the population, and the corresponding parameter of interest.
2. State the competing theories; that is, the **null** and **alternative** hypotheses.
3. State the significance level for the test.
4. Collect data, and examine your assumptions.
5. Compute test statistic, and determine the corresponding *p-value.*
6. Make a decision, and state your conclusion.

Let's do it! 10.5

State the null and alternative hypotheses that would be used to test the following statements. These statements are the researchers claim, to be stated as the alternative hypothesis. All hypotheses should be expressed in terms of μ, the population mean of interest.

- The mean age of patients in a hospital is greater than 60 years.
  - $H_o :$ $H_1 :$
- The mean caffeine content in a cup of coffee is less than 110 mg.
  - $H_o :$ $H_1 :$
- The average number of emergency room admissions per day differs from 20.
  - $H_o :$ $H_1 :$

Sampling Distribution of $\overline{X}$, the Sample Mean

If a simple random sample of size n is taken from a population with mean μ and known standard deviation σ, and

if the original population is normally distributed, i.e., if $X \sim N(\mu, \sigma^2)$, then

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right),$$

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1).$$

if the original population is not normally distributed, but if n is "large", then by the **central limit theorem**,

$$\overline{X} \text{ is approximately } \sim N\left(\mu, \frac{\sigma^2}{n}\right),$$

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \text{ is approximately } \sim N(0, 1)$$

Known Population Standard Deviation

Example *The Federal Trade Commission annually rates varieties of domestic cigarettes according to their tar, nicotine, and carbon monoxide content. The U.S. Surgeon General considers each of these substances hazardous to a smokers health. Suppose, in the past, the average carbon monoxide content has been 15 mg with a standard deviation of 4.8 mg. Have cigarettes improved on average with respect to carbon monoxide content, where lower content means better? Data from a random sample of 125 domestic brands of cigarettes given a mean CO content of $\overline{X} = 12.528$.*

Summary of the One-Sample Z test for μ: the *p-value* Approach

- We were interested in testing hypotheses about the population mean μ. The null hypothesis is $H_o : \mu = \mu_o$, where μ_o is the hypothesized value for μ. The alternative hypothesis provides the direction for the test. These hypotheses are statements about the population mean, not the sample mean.

- The data are assumed to be a random sample of size n from the population that has a normal distribution with **known** population standard deviation σ. The normality assumption is not so crucial if the sample size is large

- We base our decision about μ on the standardized sample mean value, which is

$$Z = \frac{\overline{X} - \mu_o}{\frac{\sigma}{\sqrt{n}}}.$$

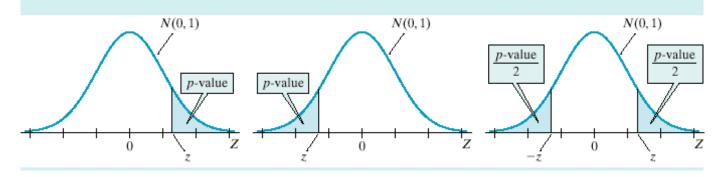 This z-score is the test statistic, and its distribution under H_0 is approximately $N(0, 1)$.

- The test statistic is the same no matter how the alternative hypothesis is expressed.

- We calculate the p-value for the test depending on how the alternative hypothesis is expressed.

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One-sided, to the right	One-sided, to the left	Two-sided
If $H_1: \mu > \mu_0$, then the p-value is the area to the right of the observed test statistic under the H_0 model.	If $H_1: \mu < \mu_0$, then the p-value is the area to the left of the observed test statistic under the H_0 model.	If $H_1: \mu \neq \mu_0$, then the p-value is the area in the two tails, outside of the observed test statistic under the H_0 model.

- If $p-value \leq \alpha$, reject H_o.

Let's do it! 10.7

At Southwestern University the average ACT mathematics score for freshmen entering college algebra is 25 with $\sigma = 1.5$. This year a random sample of freshmen will be taken to see if the average ACT math score has changed using a significance level of $\alpha = 0.05$.

- State the null and alternative hypotheses:

H_o :

H_1 :

A random sample of 30 freshman entering college algebra revealed $\overline{X} = 24.5$. Assume that the distribution of ACT scores is normal.

- What is the value of the test statistic?

- Calculate the *p-value.*

- Should the coordinator think that the average has actually changed? Explain.

Let's do it! 10.8

An electric institute publishes figures on the annual hours of usage of various home appliances. Trash compactors were stated as being used an average of 140 hours per year. However, it is hypothesized that the average usage has increased. To assess this theory, a simple random sample of 36 homes equipped with trash compactors was obtained, and the sample mean usage was 142.4 hours annually. Assume that usage is normally distributed with a population standard deviation of 9.2 hours.

Is there enough evidence to suggest that trash compactors are used on average more than 140 hours per year? State the hypotheses, calculate the test statistic, report the p-value, give your decision using a 5% significance level, and state your conclusion using a well-written sentence.

Unknown Population Standard Deviation

If the population is normal, but when the population standard deviation σ is unknown, the **test statistic** $\frac{\overline{X}-\mu_o}{s/\sqrt{n}}$, where s is the sample standard deviation, then follows a t distribution with $(n-1)$ degrees of freedom. Thus,

$$T = \frac{\overline{X} - \mu_o}{s/\sqrt{n}} \sim t^{(n-1)}.$$

The denominator $s/\sqrt{n}$ is called the **estimated standard error of the mean.**

Properties of the *t* Distribution

- The t-distribution has a symmetric bell-shaped curve centered at 0, similar to the standard normal distribution $N(0,1)$.
- The variance of the t-distribution is larger than the standard normal distribution $N(0,1)$. Therefore, the t-distribution is flatter and has heavier tails than $N(0,1)$.
- As n increases, the t-distribution approaches $N(0,1)$. I.e., as $n \rightarrow \infty, t^{(n-1)} \rightarrow N(0,1)$.

Let's do it! 10.10

A soil scientist is interested in studying the pH level in the soil for a certain field. In particular, she is going to examine a random sample of soil samples and measure their pH levels to assess if the mean field pH level is neutral (that is, equal to 7) versus the alternative hypothesis that the mean pH level is acidic (that is, less than 7).

- State the corresponding null and alternative hypothesis.

Suppose that it is reasonable to assume that the pH levels of all the possible samples that might be drawn are normally distributed. The scientist takes five randomly selected samples of soil from the field and measures the pH level in these samples. The pH levels in the sample were: 5.8, 6.3, 6.9, 6.2, 5.5.

- Find the observed sample mean pH level and the corresponding observed sample standard deviation.

- Compute the test statistic and the *p-value*.

- If the level of significance is $\alpha = 0.05$, then are the data significant?

- State your conclusions using a well-written statement.

Summary of the One-Sample *t* test for *μ*: the *p-value* Approach

- We were interested in testing hypotheses about the population mean μ. The null hypothesis is $H_o : \mu = \mu_o$, where μ_o is the hypothesized value for μ. The alternative hypothesis provides the direction for the test. These hypotheses are statements about the population mean, not the sample mean.

- The data are assumed to be a random sample of size n from the population that has a normal distribution with **unknown** population standard deviation σ. The normality assumption is not so crucial if the sample size is large

- We base our decision about μ on the standardized sample mean value, which is

$$T = \frac{\overline{X} - \mu_o}{\frac{s}{\sqrt{n}}}.$$

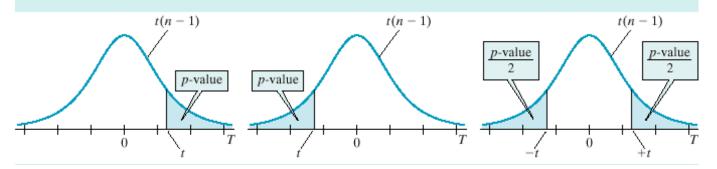 This *t*-score is the test statistic, and its distribution under H_0 is approximately $t^{(n-1)}$.

- The test statistic is the same no matter how the alternative hypothesis is expressed.

- We calculate the p-value for the test depending on how the alternative hypothesis is expressed.

One-sided, to the right	One-sided, to the left	Two-sided
If $H_1: \mu > \mu_0$, then the p-value is the area to the right of the observed test statistic under the H_0 model.	If $H_1: \mu < \mu_0$, then the p-value is the area to the left of the observed test statistic under the H_0 model.	If $H_1: \mu \neq \mu_0$, then the p-value is the area in the two tails, outside of the observed test statistic under the H_0 model.

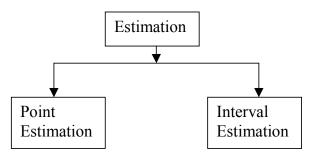- If *p-value* $\leq \alpha$, reject H_0.

Confidence Interval Estimation for μ

Question: Suppose you are interested in estimating the average amount of money a Kent State Student (population) carries. How would you find out?

Estimation Methods

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                    ┌──────────────┐
                    │  Estimation  │
                    └──────┬───────┘
              ┌────────────┴────────────┐
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      ┌──────────────┐          ┌──────────────┐
      │    Point     │          │   Interval   │
      │  Estimation  │          │  Estimation  │
      └──────────────┘          └──────────────┘
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Point Estimation

- Provides single value
  - Based on observations from 1 sample
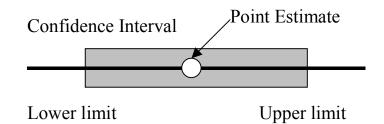- Gives no information on how close value is to the population parameter

Example *Sample mean $\overline{X} = 32$ is a point estimator of μ the average amount of money carried by a KSU student*
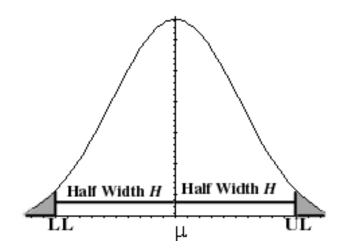
Interval Estimation

- Provides range of values
  - Based on observations from 1 sample
- Gives information about closeness to unknown population parameter
  - Stated in terms of level of confidence
  - To determine exactly, would require what information?

Example *The unknown population parameter μ, which is the average amount of money carried by a KSU student, lies between 28 and 36 with 95% confidence.*

Elements of Interval Estimation

Confidence Interval   Point Estimate

Lower limit   Upper limit

Derivation of Confidence Interval for μ (Normal Distribution)

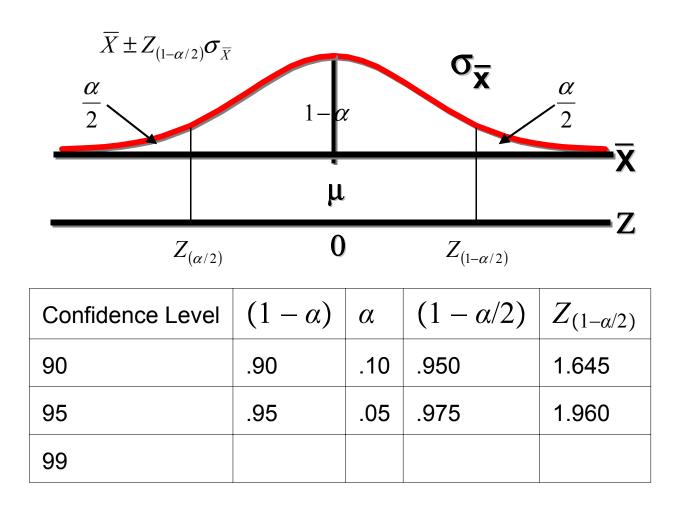Half Width H | Half Width H

LL μ UL

Confidence Interval Derivation (Continued)

1. Parameter $=$ Statistic $\pm$ Error (Half Width)
2. $\mu = \overline{X} \pm h$ (error)
3. $h = \overline{X} - \mu$ or $\overline{X} + \mu$
4. $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \dfrac{h}{\sigma/\sqrt{n}}$
5. $h = Z \times \sigma/\sqrt{n}$
6. $\mu = \overline{X} \pm Z \times \sigma/\sqrt{n}$

A $(1-\alpha)\%$ Confidence Interval (Normal Distribution)

$$\overline{X} \pm Z_{(1-\alpha/2)}\sigma_{\overline{X}}$$

$\dfrac{\alpha}{2}$

$\sigma_{\overline{X}}$

$\dfrac{\alpha}{2}$

$1-\alpha$

$\overline{X}$

μ

Z

$Z_{(\alpha/2)}$

0

$Z_{(1-\alpha/2)}$

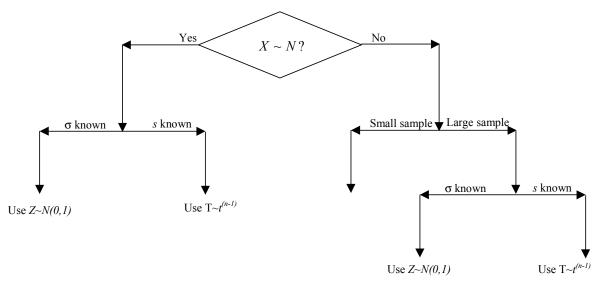| Confidence Level	$(1-\alpha)$	α	$(1-\alpha/2)$	$Z_{(1-\alpha/2)}$
90	.90	.10	.950	1.645
95	.95	.05	.975	1.960
99				

Confidence Levels

- Level of confidence denoted by $(1 - \alpha)\%$ that the unknown population parameter falls within interval.

Factors Affecting Interval Width

- Data dispersion (measured by σ)
- Sample size ($\sigma_{\overline{X}} = \sigma/\sqrt{n}$)
- Level of confidence $(1 - \alpha)$

Confidence Interval Estimates

1. Decide which distribution to use:

Yes ← $X \sim N\,?$ → No

σ known s known   Small sample   Large sample

Use Z~N(0,1)   Use T~$t^{(n-1)}$

σ known s known

Use Z~N(0,1)   Use T~$t^{(n-1)}$

2. For the standard normal $Z \sim N(0,1)$, the confidence intervals are:

Two-Sided $\overline{X} \pm Z_{(1-\alpha/2)} \dfrac{\sigma}{\sqrt{n}}$

One-Sided Upper $\mu \leq \overline{X} + Z_{(1-\alpha)} \dfrac{\sigma}{\sqrt{n}}$

One-Sided Lower $\mu \geq \overline{X} - Z_{(1-\alpha)} \dfrac{\sigma}{\sqrt{n}}$

3. For the t distribution with $(n-1)$ degrees of freedom, the confidence intervals are:

Two-Sided $\overline{X} \pm t^{(n-1)}_{(1-\alpha/2)} \frac{s}{\sqrt{n}}$

One-Sided Upper $\mu \leq \overline{X} + t^{(n-1)}_{(1-\alpha)} \frac{s}{\sqrt{n}}$

One-Sided Lower $\mu \geq \overline{X} - t^{(n-1)}_{(1-\alpha)} \frac{s}{\sqrt{n}}$

4. Remember that for very large sample sizes, $t^{(n-1)} \rightarrow N(0,1)$, and therefore you could just use the standard normal table in these cases.

Let's do it! 10.17

A beverage dispensing machine is calibrated so that the amount of beverage dispensed is approximately normally distributed with a population standard deviation of 0.15 deciliters (dL).

- Compute a 95% confidence interval for the mean amount of beverage dispensed by this machine based on a random sample of 36 drinks dispensing an average of 2.25 dL.

- Would a 90% confidence interval be wider or narrower than the interval above.

- How large of a sample would you need if you want the width of the 95% confidence interval to be 0.04?
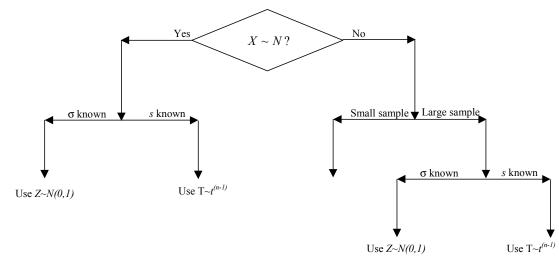
Let's do it! 10.18

A restaurant owner believed that customer spending was below the usual spending level. The owner takes a simple random sample of 26 receipts from the previous weeks receipts. The amount spent per customer served (in dollars) was recorded and some summary measures are provided:

$n = 26, \overline{X} = 10.44, s^2 = 7.968, \min = 4, \max = 16, Q1$

- Assuming that customer spending is approximately normally distributed, compute a 90% confidence interval for the mean amount of money spent per customer served.

- Interpret what the 90% confidence interval means.

Summary: Chapter 10

1. First decide which table to use

2. Calculate $S_{\overline{X}}$ or $\sigma_{\overline{X}}$ as the case may be. Remember $S_{\overline{X}} = \frac{s}{\sqrt{n}}$ and $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$. $S_{\overline{X}}$ is called the standard error of the sample mean.

3. Remember that the level of confidence is given by $1 - \alpha$.

4. In hypothesis testing, the test statistic is given by $Z^* = \frac{\overline{X} - \mu_o}{\sigma_{\overline{X}}}$ or $t^* = \frac{\overline{X} - \mu_o}{S_{\overline{X}}}$, as the case may be.

Tests Based on P-Values:

Reject H_o if p-value $< \alpha$, and do not reject H_o if p-value $\geq \alpha$.

Types of Errors:

Type I error $\alpha = P(\text{Conclude } H_1 | H_o \text{ is true})$

Type II error $\beta = P(\text{Conclude } H_0 | H_1 \text{ is true})$

Procedures for Table II:

Confidence Intervals:

Two-Sided $\overline{X} \pm Z_{(1-\alpha/2)} \sigma_{\overline{X}}$

One-Sided Upper $\mu \leq \overline{X} + Z_{(1-\alpha)} \sigma_{\overline{X}}$

One-Sided Lower $\mu \geq \overline{X} - Z_{(1-\alpha)} \sigma_{\overline{X}}$

Hypothesis Testing:

Two-Sided: $H_o : \mu = \mu_o \,;\, H_1 : \mu \neq \mu_o$

Reject H_o if $Z^* > Z_{(1-\alpha/2)}$ or $Z^* < -Z_{(1-\alpha/2)}$;

Conversely, Do not reject H_o if $-Z_{(1-\alpha/2)} \leq Z^* \leq Z_{(1-\alpha/2)}$

One-Sided Upper Tail: $H_o : \mu \leq \mu_o \,;\, H_1 : \mu > \mu_o$

Reject H_o if $Z^* > Z_{(1-\alpha)}$; Do not reject H_o if $Z^* \leq Z_{(1-\alpha)}$

One-Sided Lower Tail: $H_o : \mu \geq \mu_o \,;\, H_1 : \mu < \mu_o$

Reject H_o if $Z^* < -Z_{(1-\alpha)}$; Do not reject H_o if $Z^* \geq -Z_{(1-\alpha)}$

P-Values (assume that the mean under H_o is true when calculating this):

Two-Sided $2 \times \min[\text{P}(Z \geq Z^*, Z \leq Z^*)]$

One-Sided Upper Tail $\text{P}(Z \geq Z^*)$

One-Sided Lower Tail $P(Z \leq Z^*)$

Procedure for Table IV:

Confidence Intervals:

Two-Sided $\overline{X} \pm t_{(n-1;1-\alpha/2)} S_{\overline{X}}$

One-Sided Upper $\mu \leq \overline{X} + t_{(n-1;1-\alpha)} S_{\overline{X}}$

One-Sided Lower $\mu \geq \overline{X} - t_{(n-1;1-\alpha)} S_{\overline{X}}$

Hypothesis Testing:

Two-Sided: $H_o : \mu = \mu_o \; ; H_1 : \mu \neq \mu_o$

Reject H_o if $t^* > t_{(n-1;1-\alpha/2)}$ or $t^* < -t_{(n-1;1-\alpha/2)}$;

Conversely, Do not reject H_o if

$-t_{(n-1;1-\alpha/2)} \leq t^* \leq t_{(n-1;1-\alpha/2)}$

One-Sided Upper Tail: $H_o : \mu \leq \mu_o \; ; H_1 : \mu > \mu_o$

Reject H_o if $t^* > t_{(n-1;1-\alpha)}$; Do not reject H_o if

$t^* \leq t_{(n-1;1-\alpha)}$

One-Sided Lower Tail: $H_o : \mu \geq \mu_o \; ; H_1 : \mu < \mu_o$

Reject H_o if $t^* < -t_{(n-1;1-\alpha)}$; Do not reject H_o if

$t^* \geq -t_{(n-1;1-\alpha)}$

P-Values (assume that the mean under H_o is true when calculating this):

Two-Sided $2 \times \min[P(t \geq t^*, t \leq t^*)]$

One-Sided Upper Tail $P(t \geq t^*)$

One-Sided Lower Tail $P(t \leq t^*)$