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Content (in the order sections were covered from the book):



# Chapter 6 – Higher-Degree Polynomial Functions

## Algebra Toolbox

**Polynomials with a special name:**

- **Monomials (also called terms): 3, -0.5x,**
- **Binomials (two terms) :** $4x+5$, $x^2+2$, $12x^4+6x$,
- **Trinomials (three terms):** $x^4-6x^2+8$, $-6x^4-10x^3+4x^2$

**Generic form of the polynomial:** $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$n$ **is the degree of the polynomial** and $a_n$ **is the leading coefficient.**

**Examples:**

a) $-6x^4-10x^3+4x^2$ **is a fourth-degree polynomial with the leading coefficient -6.**
b) $3$ **is a zero degree polynomial** ($3*x^0$) **with the leading coefficient 3.**
c) $4x+5$ **is a first-degree polynomial** ($4*x^1$) **with the leading coefficient 4.**

**Example 2: Factor** $-6x^4-10x^3+4x^2$

**Example 3: Factor** $x^4-6x^2+8$

**Rational expressions (the quotient of two polynomials):** $\frac{3x}{3x+6}$, $\frac{2x^2-8}{x+2}$

**Example 4: Simplify a.** $\frac{3x}{3x+6}$ **b.** $\frac{2x^2-8}{x+2}$

**Do problems 11-15 in toolbox exercises.**

## Section 6.1 Higher-Degree Polynomial Functions

**So far we used models represented by linear ( $ax + b$ ) or quadratic ($ax^2 + bx + c$).**

**They are first and second-degree polynomial functions.**

**Now we'll work with higher-degree polynomial functions. Some examples are:**

$y = 3x^4 - x^3 + 2$ $\qquad$ $y = x^3 - 27x$

**Cubic graph:**
**Graph cubic** $y = x^3 - 27x$ **Use the window: [-10, 10] by [-100, 100]**
**Find the local maximum and local minimum if possible.**
**What are the turning points?**

**Quartic graph:**
**Graph quartic** $y = 3x^4 - 4x^3$ **Use the window: [-2, 2] by [-3, 3]**
**Find the local maximum and local minimum if possible.**
**What are the turning points?**
**Minimums: [0.999, -1],**

**Review Table 6.3 on p. 434**
**The graph of polynomial functions has at most $n - 1$ turning points and $n$ x-intercepts.**
**The ends of polynomial graphs:**

- **with an odd degree are "one end opening up and other end opening down".**
- **with even degree are "both ends opening up or both ends opening down"**

## Section 6.2 Modeling with Cubic and Quartic Functions

**Example 4 The percent of the US population that was foreign-born.**

a. **Create a scatter plot for the data using the number of years after 1900 as the input**
b. **Create a cubic function to model the data and graph the function on the same axes**
c. **Find a quartic function to model the data and graph the function on the same axes**
d. **Use the graphs from (b) and (c) to determine which function is the better fit**

| year | 1900 | 1910 | 1920 | 1930 | 1940 | 1950 | 1960 | 1970 | 1980 | 1990 | 2000 | 2005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| % | 13.6 | 14.7 | 13.2 | 11.6 | 8.8 | 6.9 | 5.4 | 4.8 | 6.2 | 8.0 | 10.4 | 11.7 |
| Compute 1-st, 2-nd , 3-rd and 4-th differences for first 6 rows: | | | | | | | | | | | | |
| 1-st | | 1.1 | -1.5 | -1.6 | -2.8 | -1.9 | | | | | | |
| 2-nd | | | -2.6 | -0.1 | -1.2 | 0.9 | | | | | | |
| 3-rd | | | | 2.5 | -1.1 | 2.1 | | | | | | |
| 4-th | | | | | 3.6 | 3.2 | | | | | | |

**Do 6.2:35**

## Section 6.3 Solution of Polynomial Equations

### Solving by Factoring

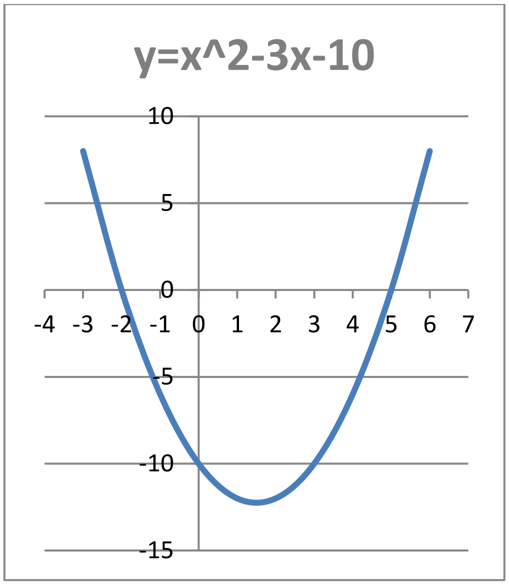
**Solve** $0 = x^2 - 3x - 10$

a. **by factoring**
b. **graphically by finding zeroes for:** $f(x) = x^2 - 3x - 10$
   **Note that zeroes of the function (x- intercepts) are solutions for the equation.**

**The solution:**

**a. The solution by factoring: (x-5)(x-2)=0 The solutions are x=5 or x=2**

b. From the graph, we can conclude that 5 and 2 are zeroes (x-intercepts) for
$f(x) = x^2 - 3x - 10$




**A similar approach can be applied to higher-degree polynomials:**

| |
|---|
| **If higher degree polynomials can be factored, each factor represents a solution for the corresponding equation.** |

**Example 3 For** $y = 120x^2 - 20x^3$ **(photosynthesis example on p 461)**

*a. **What kind of graph do we expect according to the table on p. 434***
b. **Graph this function so that you can see two turning points.**
c. **Use factoring to find x-intercepts of the graph.**
d. **What nonnegative values of x will give positive y-values?**
e. **Graph the function with the viewing window that shows only graph for x-values from d.**

The solution:

a. Both ends of the graph should be falling.
b. Use window [-10, 10] by [-1000, 1000]
c. $y = 20x^2(6 - x)$ the solutions are x=0 or 6-x=0 (x-6)
d. $0 < x < 6$
e. First find local maximum (4, 640) and then Use window [0, 6] by [0, 700]

## Solving Using Factoring by Grouping

**When a higher-degree polynomial has 4 terms it sometimes can be factored by grouping.**

**Example 4. The total cost of producing a product is given by the function:**

$C(x) = x^3 - 12x^2 + 3x + 9$ **thousand dollars**

**where x is the number of hundreds of units produced.**

**How many units must be produced to give a total cost of \$45,000?**

**Algebraic solution:**

$45 = x^3 - 12x^2 + 3x + 9$

$0 = x^3 - 12x^2 + 3x - 36$

**Factor** $0 = x^3 - 12x^2 + 3x - 36$

$0 = x^2(x - 12) + 3(x - 12)$

$0 = (x - 12)(x^2 + 3)$ **a.** $x - 12 = 0$

$x = 12$ **is the only solution (12 hundreds of units)**

**b.** $x^2 + 3 = 0$ **does not have a solution in real numbers**

**Producing 1,200 units results in a total cost of \$45,000.**

**ICE: 6.3:2 Solve** $(3x + 1)(2x - 1)(x + 5) = 0$

## The Root Method

**The root method is used when there is only one term with a variable raised to the power.**

**For** $x^n = C$ **the solutions in real numbers are:**

$x = \sqrt[n]{C}$ **if $n$ is odd**

$x = \pm\sqrt[n]{C}$ **if $n$ is even and $C \geq 0$** (for $C < 0$ even root is not defined in real numbers)

***Observe that for odd roots there is ONLY ONE solution while for even roots there are two.***

Steps to compute $3 * \sqrt[4]{16}$ on the graphing calculator:

1. Enter 3*4
2. [MATH] and then select 5 ($\sqrt[x]{\ }$). That produces $3 * 4\sqrt[x]{\ }$ on the display
3. Enter 16 (the display will look like: $3 * 4\sqrt[x]{16}$ )
4. [ENTER] will produce the result (6)

Example 5.

a. $x^3 = 125$ b. $5x^4 = 80$ c. $4x^2 = 18$

ICE: 6.3:18 Solve $2x^4 - 162 = 0$

Example 6 The future value of $10,000 invested for 4 years at the rate *r* is given by the formula: $S = 10,000(1 + r)^4$

Find rate *r*, as a percent, for which the future value is $14,641.

## Estimating Solutions with Technology

When higher-degree equations cannot be solved by factoring, the graphical solution must be used.

Example 7 The annual number of arrests for crime per 100,000 juveniles from 10 to 17 years of age can be modeled by the function:

$$f(x) = -0.357x^3 + 9.417x^2 - 51.852x + 361.208$$

where x is the number of years after 1980. The number of arrests peaked in 1994 and then decreased. Use this model to estimate graphically the year after 1980 in which the number of arrests fell to 234 per 100,000 juveniles.

# Graphs of Rational Functions

**$P(x)$ and $Q(x)$ are polynomials.**

**Rational function :** $f(x) = \frac{P(x)}{Q(x)}$ **is defined only for $Q(x) \neq 0$**

**Example: Graph $f(x) = \frac{1}{x}$**

Observe that $P(x)$=1 and $Q(x) = x$ and that y grow to $\mp\infty$ as x approaches the 0. The line y=0 is asymptote for $f(x)$ .

**The asymptote of a function is a line such that the distance between the function and the line approaches zero as they tend to infinity.**

**Note that although some functions may have asymptotes you will not always be asked to find or interpret them (read the questions carefully and answer only what you are asked about).**

**For the rational function:** $f(x) = \frac{P(x)}{Q(x)}$ **where $Q(x) \neq 0$**

**Vertical asymptote occurs in x-values where $Q(x) = 0$ and $P(x) \neq 0$**

**Vertical asymptotes should not be visible on the graph.**
**For some graphs, older calculators show vertical asymptotes as lines when [MODE] is set to Connected.**

**Vertical asymptote for $f(x) = \frac{1}{x}$ is in x=0**

Observe that $Q(x) = x$

Example 1 Find the vertical asymptote for $f(x) = \frac{x+1}{(x+2)^2}$

Observe that $Q(x) = (x+2)^2$ $P(x) = x + 1$

Example 2. Function of the cost of removing $p\%$ of pollution is $C(p) = \frac{800p}{100-p}$

a. Graph the function on [-100, 200] by [-4000, 4000]

b. Does this graph have a vertical asymptote on this window? Where?

c. For what values of $p$ does this function serve as a model for the cost of removing particulate pollution?

d. Use the information from (c) to graph the model on the appropriate window

e. What does the part of the graph near the vertical asymptote tell us about the cost of removing particulate pollution?

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Some graphs have horizontal asymptotes (the graph appears horizontal one or both ends):

$f(x) = \frac{1}{x}$ and $f(x) = \frac{x+1}{(x+2)^2}$

For the rational function: $f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots b_0}$

Horizontal asymptote:

1. If $n < m$ horizontal asymptote is $y = 0$
2. If $n = m$ horizontal asymptote is $y = \frac{a_n}{b_m}$
3. If $n > m$ there is no horizontal asymptote

Graph $f(x) = \frac{6-5x}{x^2}$ Observe that the graph can cross horizontal asymptote.

Example 3. Daily average cost (in $hundred) is represented by: $\overline{C} = \frac{25+13x+x^2}{x}$

$x$ is a number of produced units (golf carts)

a. Graph the function in the windows [-20, 20] by [-30, 50]
b. Does the graph in (a) have a horizontal asymptote?
c. Graph the function on the window [0, 20] by [0,50]
d. Does the graph using a window in (a) or (c) better model the average cost function? Why?
e. Find the minimum daily average cost and number of golf carts that give the minimum daily average cost.

## Algebraic and Graphical Solution of Rational Equations

**Steps to solve a rational equation:**

1. Multiply both sides with LCD (least common denominator)
2. Solve the resulting polynomial
3. Check the solution in the original equation

**Example 4** Solve $x^2 + \frac{x}{x-1} = x + \frac{x^3}{x-1}$

**Example 5** Monthly sales (in thousands of dollars) is represented by $y = \frac{300x}{15+x}$

where $x$ are monthly advertising expenses (in thousands of dollars). Determine the amount of money that must be spent on advertising to generate $100,000 in sales.

a. Solve algebraically
b. Solve graphically

## Algebraic Solution of Linear Inequalities

**When multiplying (dividing) inequality by a negative number, the direction of the inequality symbol must be reversed.**

For example : 3 > 2
If inequality is multiplied by -1 : (-1)*3 **<** (-1) *2 this is because -3 < -2

**The solution of linear inequalities is typically an interval.**

| | Equality | Inequality |
|---|---|---|
| **A problem (example 1)** | $3x - \frac{1}{3} = -4 + x$ | $3x - \frac{1}{3} \leq -4 + x$ |
| **Multiply both sides by the least common denominator** | $3(3x - \frac{1}{3}) = 3(-4 + x)$ | $3(3x - \frac{1}{3}) \leq 3(-4 + x)$ |
| **Remove any parentheses** | $9x - 1 = -12 + 3x$ | $9x - 1 \leq -12 + 3x$ |
| **Get variable on one side and all other items on another** | $6x = -11$ | $6x \leq -11$ |
| **Divide both sides by the coefficient of the variable. If the coefficient is negative reverse the inequality symbol.** | $x = -\frac{11}{6}$ | $x \leq -\frac{11}{6}$ or $(-\infty, -\frac{11}{6}]$ |
| **Check the solution by substitution or with a graphing utility.** | $3 * \left(-\frac{11}{6}\right) - \frac{1}{3} = -4 + (-\frac{11}{6})$ | $3 * \left(-\frac{11}{6}\right) - \frac{1}{3} \leq -4 + (-\frac{11}{6})$ |

| | Equality | Inequality |
|---|---|---|
| **A problem (2.4:2)** | $2x + 6 = 4x + 5$ | $2x + 6 < 4x + 5$ |
| **Multiply both sides by the least common denominator** | Nothing to do | **Nothing to do** |
| **Remove any parentheses** | Nothing to do | **Nothing to do** |

| | | |
|---|---|---|
| Get variable on one side and all other items on another | $-2x = -1$ | $-2x < -1$ |
| Divide both sides by the coefficient of the variable. If the coefficient is negative reverse the inequality symbol. | $x = \frac{1}{2}$ | $x > \frac{1}{2}$ or $(\frac{1}{2}, +\infty)$ |
| Check the solution by substitution or with a graphing utility. | $2 * \frac{1}{2} + 6 = 4 * \frac{1}{2} + 5$ | $2 * \frac{1}{2} + 6 < 4 * \frac{1}{2} + 5$ |

## Graphical Solution of Linear Inequalities

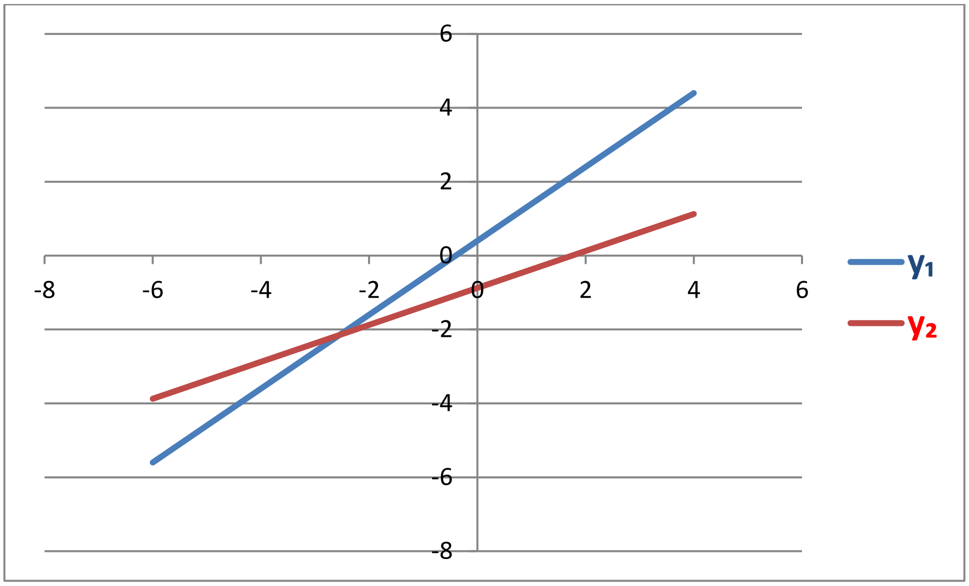
**The steps are the same as when solving equations graphically only the solution, in this case, is an interval and we must be aware of the relationship between $y_1$ and $y_2$**

a. **Intersection method – set the left side of the inequality as $y_1$ and right as $y_2$**
   - **find the intersection**
b. **x-intercept method – move all terms to the left side leaving 0 on the right side**
   - **graph left side as $y_1$ and find the x-intercept**

   **Observe that x-intercept is actually the intersection of $y_1$ and $y_2 = 0$**

**Example 4 Solve** $\frac{5x+2}{5} \geq \frac{4x-7}{8}$ **using the intersection of graphs method.**

$y_1 = \frac{5x+2}{5}$ $\qquad$ $y_2 = \frac{4x-7}{8}$




**The intersection is (-2.55, -2.15).**

$y_1$ **is greater than** $y_2$ **for** $x \geq -2.55$ **or in interval notation [-2.55, ∞)**

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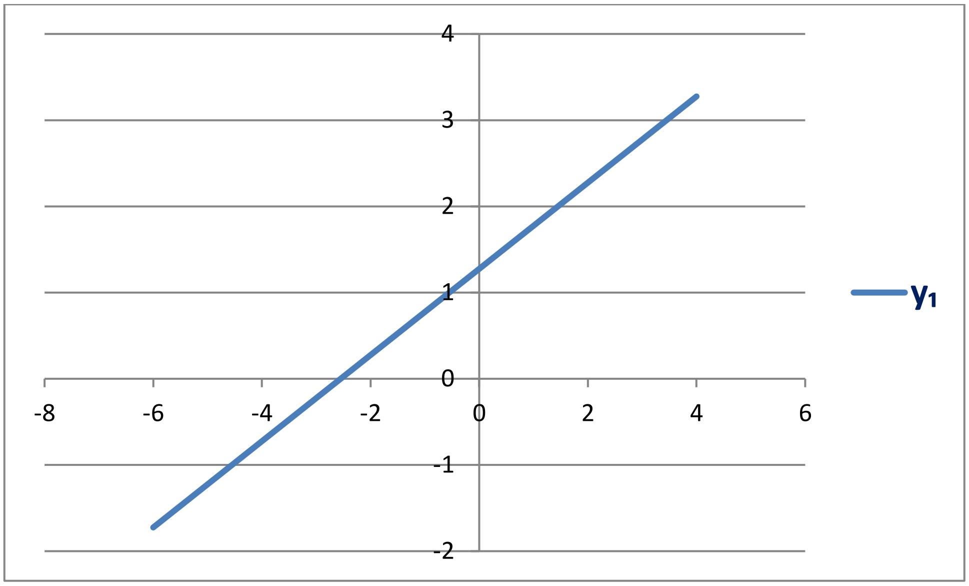
**The same problem can be solved using the x-intercept method.**

$$\frac{5x+2}{5} \geq \frac{4x-7}{8}$$

$$\frac{5x+2}{5} - \frac{4x-7}{8} \geq 0$$




**x-intercept is (-2.55, 0)**

**The solution is $x \geq -2.55$ or in interval notation [-2.55, ∞)**

**2.4:10** $\frac{2(x-4)}{3} \geq \frac{3x}{5} - 8$

**a. Solve algebraically and graphically.**

**b. Draw a number line of the solution.**

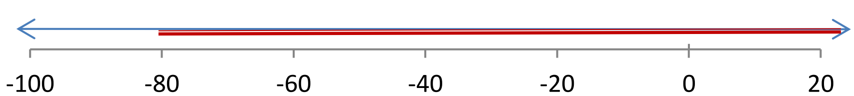
**a. LCD is 3*5=15** $15 * \left(\frac{2(x-4)}{3}\right) \geq 15 * \left(\frac{3x}{5} - 8\right)$

$$5 * (2x - 8) \geq 15 * \frac{3x}{5} - 15 * 8$$

$$10x - 40 \geq 9x - 120$$

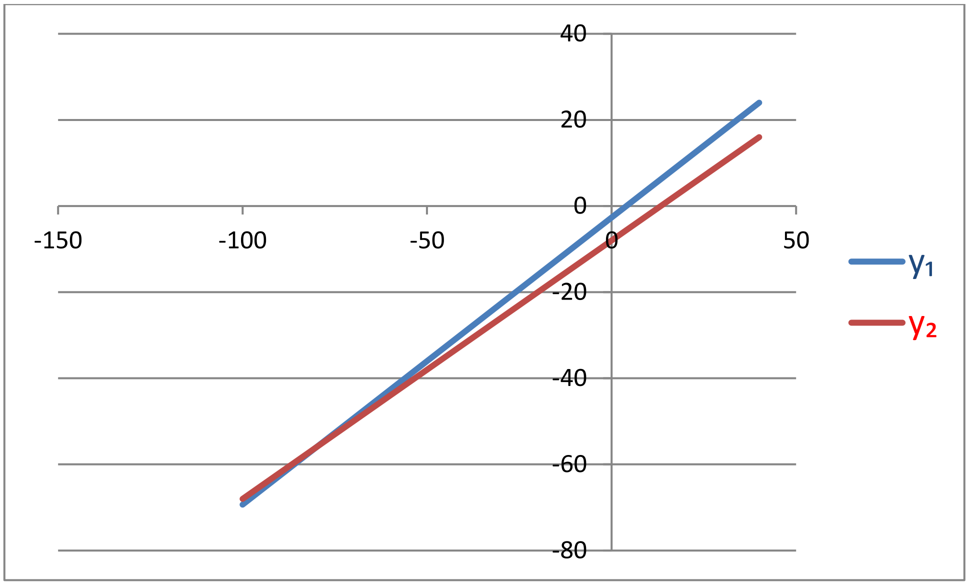
$x \geq -80$ **algebraic solution**

**[80,∞)**




**Graphical solution:** $\frac{2(x-4)}{3} \geq \frac{3x}{5} - 8$

**Intersection method:** $y_1 = \frac{2(x-4)}{3}$ $\quad y_2 = \frac{3x}{5} - 8$




**Review 2.4:11**

## Double Inequalities

**Double inequalities are two inequalities connected by the word *and* or *or*.**

**Example for *and*:** $3 < x$ ***and*** $x \leq 6$ $\quad 3 < x \leq 6$ **(shorthand for *and*)**

**Example for *or* :** $x \leq 2$ ***or*** $x \geq 4$ **(no shorthand for *or*)**

**Example 6** **To receive B grade, the final test score, represented by x, must satisfy**

$$80 \leq \frac{90+88+93+85+x}{5} < 90$$

**2.4:23 Solve** $3x + 1 < -7$ ***and*** $2x - 5 > 6$

**2.4:26 Solve** $\frac{1}{2}x - 3 < 5$ ***or*** $\frac{2}{5}x - 5 > 6x$

**ICE: 2.4:24 Solve** $6x - 2 \leq -5$ ***or*** $3x + 4 > 9$

## Section 6.6 Polynomial and Rational Inequalities

**When solving algebraically, one side of inequality must be equal to 0 (this is the same as when solving equations earlier).**

## Polynomial Inequalities

**6.6:5** $(x-1)(x-3)(x+1) \geq 0$

| A sign diagram | | | | |
|---|---|---|---|---|
| | $x \leq -1$ | $-1 \leq x \leq 1$ | $1 \leq x \leq 3$ | $x \geq 3$ |
| $(x-1)$ | - | - | + | + |
| $(x-3)$ | - | - | - | + |
| $(x+1)$ | - | + | + | + |
| **The product** | - | + | - | + |

**X values from the intervals** $-1 \leq x \leq 1$ **and** $x \geq 3$ **make the inequality true so these two intervals are.**

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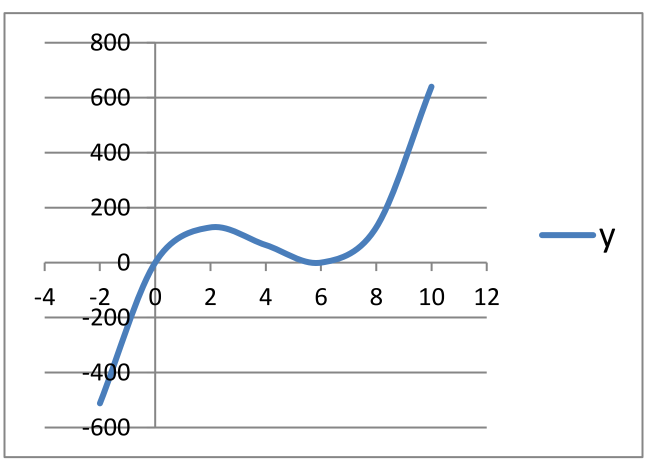
**Solve** $144x - 48x^2 + 4x^3 > 0$




**Graphical solution:** $y = 144x - 48x^2 + 4x^3$

**x-intercepts are 0 and 6**

$y > 0$ **for** $0 < x < 6$ **and** $x > 6$

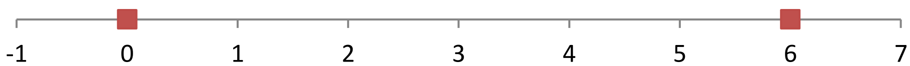
**Interval notation:** $(0, 6)$ **and** $(6, \infty)$

**Algebraic solution by factoring:**

1. **Solve an equation to find zeroes**

$$144x - 48x^2 + 4x^3 = 0$$
$$4x^3 - 48x^2 + 144x = 0$$
$$4x(x^2 - 12x + 36) = 0$$
$4x(x-6)^2 = 0$ the solutions are: $x = 0$ or $x = 6$ $(x - 6 = 0)$

2. Zeroes divide number line into intervals: x <0 0 < x < 6 x > 6




3. These intervals are checked against original inequality. Signs for each factor are entered into a table (table is called sign diagram) and product sign is evaluated.

| A sign diagram | | | |
|---|---|---|---|
| | x < 0 | 0 < x < 6 | x > 6 |
| x | - | + | + |
| $(x-6)^2$ | + | + | + |
| The product | - | + | + |

## Rational Inequalities

Example 2 The average cost per set for the television sets is

$$\overline{C} = \frac{5000+80x+x^2}{x}$$

where x is the number of hundreds of units produced.

Find the number of television sets that must be produced to keep the average Cost to at most $590.

Solve algebraically:

$$\frac{5000+80x+x^2}{x} \le 590$$

$$\frac{5000+80x+x^2}{x} - 590 \le 0$$

$$\frac{5000+80x+x^2-590x}{x} \le 0$$

$$\frac{x^2-510x+5000}{x} \leq 0 \qquad \frac{(x-10)(x-500)}{x} \leq 0$$

$$\frac{(x-10)(x-500)}{x} = 0$$ **Zeroes are 10 and 500.**


**The intervals to consider are :** $x \leq 10 \qquad 10 \leq x \leq 500 \qquad x \geq 500$

**Since negative values for x do not make sense for that reason first interval is $0 < x \leq 10$ instead of $\leq 10$ .**

**Observe that the interval is NOT $0 \leq x \leq 10$ because the rational expression is not defined so x=0 cannot be included in the inequality solution.**

| **A sign diagram** | | | |
|---|---|---|---|
| | $0 < x \leq 10$ | $10 \leq x \leq 500$ | $x \geq 500$ |
| **x** | + | + | + |
| **(x-10)** | - | + | + |
| **(x-500)** | - | - | + |
| **The product** | + | - | + |

**The original inequality is** $\frac{x^2-510x+5000}{x} \leq 0$ **so the product needs to be less than or equal to 0 and that happens only for the interval** $10 \leq x \leq 500$

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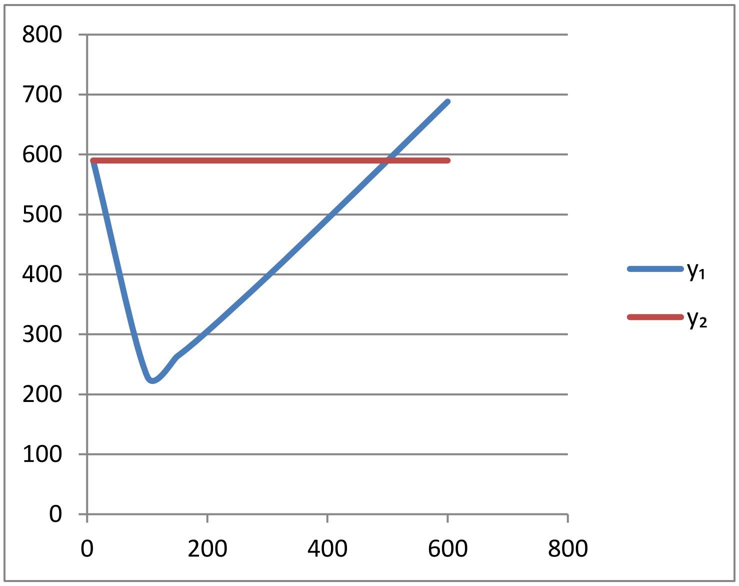
**Solve graphically:**

$$\frac{5000+80x+x^2}{x} \leq 590$$

Intersection method: $y_1 = \frac{5000+80x+x^2}{x}$ $\quad y_2 = 590$




Intersections are (10, 590) and (500, 590)

$y_1$ is below or equal 590 for x values in [10, 500]

or

$10 \le x \le 500$

ICE: 6.6:9 Solve both graphically and algebraically.