2009 Stark County
High School
Mathematics Challenge

ANSWER KEY

May 02, 2009

INSTRUCTIONS

You will have **One Hour** to answer each of the following ten questions (in Part I) to the best of your ability. Each question is worth ten points. Please show all of your work. Partial credit will be awarded for sound reasoning and partial solutions. There will be no talking during the test. If you have any questions as to the wording of a problem, please raise your hand and the proctor will assist you. No other questions will be answered.

All tests will be collected promptly at the end of the hour. If you finish early, please close your test booklet and quietly exit the room. No one will be re-admitted to the testing room once they have left; i.e. no bathroom breaks will be allowed. Anyone that is deemed by the proctor to be talking, or otherwise cheating will be asked to leave immediately and will receive a score of zero for the entire competition. You have a short snack break after Part I is completed. Part II will be similar in format, but single answers need to be put in the answer card provided. You will have 40 minutes to complete Part II.

Good Luck, and remember to keep your ε's positive.

Part I

Test 1

Problem I.1 *What is the sum of the first fifteen terms of the following sequence?*

$$\{3, 7, 11, 15, 19, \ldots\}$$

Solution:

The sequence is arithmetic with $a = 3, d = 4$. Calculate $a_n = 3 + (15 - 1)(4) = 59$. Sum can be obtained by Gauss's trick as,

$$S = (62) \cdot \frac{15}{2} = 465.$$

Any other reasonable method with correct answer gets credit.

Problem I.2 *What is the largest value of x, such that 3^x divides $30!$? [Factorial notation is defined by $n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$]*

Solution:

Consider all multiples of 3 that appear in the sequence of $1, 2, \cdots, 29, 30$. They are,

$$
\begin{aligned}
3 &= 3 \\
6 &= 2 \cdot 3 \\
9 &= 3^2 \\
12 &= 2^2 \cdot 3 \\
15 &= 3 \cdot 5 \\
18 &= 2 \cdot 3^2 \\
21 &= 3 \cdot 7 \\
24 &= 2^3 \cdot 3 \\
27 &= 3^3 \\
30 &= 2 \cdot 3 \cdot 5
\end{aligned}
$$

The largest value of x such that 3^x divides $30!$ is $x = 1 + 1 + 2 + 1 + 1 + 2 + 1 + 1 + 3 + 1 = 14$.

No partial credit.

Problem I.3 *Let* $f(x) = \frac{x^4-1}{x^4+2x^2+1}$ *and* $g(x) = \sqrt{x}$*. Find and* ***simplify*** $(f \circ g)(x)$*.*

Solution:
Observe that $f(x) = \frac{x^4-1}{x^4+2x^2+1} = \frac{(x^2+1)(x^2-1)}{(x^2+1)^2} = \frac{x^2-1}{x^2+1}$

$$(f \circ g)(x) = f(\sqrt{x}) = \frac{x-1}{x+1}.$$

No partial credit.

Problem I.4 *How many of the positive factors of the number* $36,000,000$ *are* ***not*** *perfect squares? [A positive integer is a perfect square if it can be written as the square of another positive integer, for example* $4 = 2^2, 9 = 3^2$ *etc.]*

Solution:

Observe that $36,000,000 = 2^8 \cdot 3^2 \cdot 5^6$, every positive factor must be of the form $2^p \cdot 3^q \cdot 5^r$, where $p \in \{0, 1, 2, \cdots, 8\}$, $q \in \{0, 1, 2\}$, and $r \in \{0, 1, 2, \cdots, 6\}$. Therefore, there are $9 \times 3 \times 7 = 189$ factors.

The factors which are perfect squares have the form $(2^2)^j \cdot (3^2)^k \cdot (5^2)^l$, where $j \in \{0, 1, 2, 3, 4\}$, $k \in \{0, 1\}$, and $l \in \{0, 1, 2, 3\}$, so there are $5 \times 2 \times 4 = 40$ of these.

Thus, the number of factors that are **not** perfect squares is $= 189 - 40 = 149$.

No partial credit.

Problem I.5 *Show that the polynomial* $f(x) = x^3 + (2+a)x^2 + (5+2a)x + 5a$*, where* a *is a constant real number, has one real root and two complex roots. Find these three roots. [HINT: Guess a solution and do synthetic or long division]*

Solution:
Observe that $x = -a$ is a root since $f(-a) = 0$.
Using synthetic division or long division, we can write

$$\begin{aligned}
f(x) &= (x+a)(x^2+2x+5) \\
\text{Solutions to } x^2+2x+5 &= 0 \text{ are:} \\
x &= \frac{-2 \pm \sqrt{4-4(1)(5)}}{2} \\
&= \frac{-2 \pm \sqrt{-16}}{2} \\
&= \frac{-2 \pm 4i}{2} = -1 \pm 2i \\
&\therefore \; x = -a, -1 \pm 2i.
\end{aligned}$$

Give partial credit for getting one of the roots correct.

Problem I.6 *The function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies the condition*

$$mf(x-1) + nf(-x) = 2|x| + 1.$$

If $f(-2) = 5$ and $f(1) = 1$, then what does $m + n$ equal? Here $\mathbb{R}$ refers to the set of real numbers.

Solution:

$$\begin{aligned}
mf(x-1) + nf(-x) &= 2|x| + 1 && (1)\\
\text{Substitute } x &= -1 \text{ in } (1): \\
mf(-2) + nf(1) &= 2|-1| + 1 \\
5m + n &= 3 && (2)\\
\text{Substitute } x &= 2 \text{ in } (1): \\
mf(1) + nf(-2) &= 2|2| + 1 \\
m + 5n &= 5 && (3)\\
\text{Add (2) and (3)} &: \\
6(m+n) &= 8 \\
m + n &= \frac{4}{3}.
\end{aligned}$$

Give partial credit if they had the right idea but made a mistake in the algebra.

Problem I.7 *Prove the following trigonometric identity.*

$$\frac{1}{\sin^2(\theta)+\sin^2(\theta)\cos(2\theta)}+\frac{\cos(2\theta)-\csc^2(\theta)\cos(2\theta)-1}{1+\cos(2\theta)}\equiv 1$$

Solution:

$$\begin{aligned}
\text{LHS} &= \frac{1}{\sin^2(\theta)[1+\cos(2\theta)]}+\frac{\cos(2\theta)-\csc^2(\theta)\cos(2\theta)-1}{[1+\cos(2\theta)]}\\
&= \frac{\csc^2(\theta)}{[1+\cos(2\theta)]}+\frac{\cos(2\theta)-\csc^2(\theta)\cos(2\theta)-1}{[1+\cos(2\theta)]}\\
&= \frac{\csc^2(\theta)[1-\cos(2\theta)]-[1-\cos(2\theta)]}{[1+\cos(2\theta)]}\\
&= \frac{[1-\cos(2\theta)][\csc^2(\theta)-1]}{[1+\cos(2\theta)]}\\
&= \frac{2\sin^2(\theta)\cdot\cot^2(\theta)}{2\cos^2(\theta)}=1=\ \text{RHS.}
\end{aligned}$$

Give credit to any reasonable method of proof.

Problem I.8 *A regular tetrahedron has edges 36 units in length. What is the altitude of the tetrahedron? Give your answer in simplest radical form $x\sqrt{y}$. [A regular tetrahedron has four equilateral triangles]*

Solution:

A regular tetrahedron is made up of equilateral triangles. Call the center of this triangle O. If each side of one of these triangles is 36 then its altitude is $18\sqrt{3}$ (use Pythagorean Theorem).

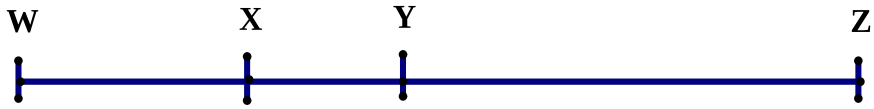
Dividing this triangle into six congruent 30-60-90 right triangles. Let h be the distance from the base of one of these six triangles to the center O (of the big triangle). Then

$$\begin{aligned} \tan 30^\circ &= \frac{h}{18} \\ h &= 18\tan 30^\circ = 18 \cdot \frac{\sqrt{3}}{3} = 6\sqrt{3}. \end{aligned}$$

If a is the altitude of the tetrahedron, then

$$\begin{aligned} a^2 + (6\sqrt{3})^2 &= (18\sqrt{3})^2 \\ a^2 + 108 &= 972 \\ a^2 &= 864 \\ a &= 12\sqrt{6}. \end{aligned}$$

Partial credit may be given for a resonable attempt towards the correct answer.

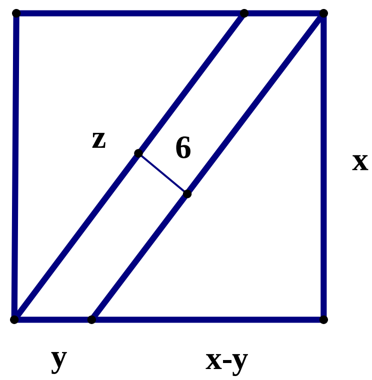


Problem I.9 *The length of WY is 21. The length of XZ is 26. The length of YZ is twice that of WX. Find the length of XY.*

Solution:
Let $WX = x$. Then $YZ = 2x$.

$$\begin{aligned} WY + YZ &= WX + XZ \\ 21 + 2x &= x + 26 \\ x &= 5 \\ &\therefore\ XY = 21 - 5 = 16. \end{aligned}$$

Give partial credit for correct value of x.



Problem I.10 *A square is divided into three pieces of equal area by two parallel cuts, as shown. The distance between the parallel lines is 6 inches. What is the area of the square?*

Solution:

Let the length of a side of the square be x and the length of the short segment on the side be y, as shown. The diagonal length z is given by $z = \sqrt{x^2 + (x-y)^2}$. Also since the area of the parallelogram is $\frac{1}{3}$ the area of the square, $xy = \frac{x^2}{3}$, so that $y = \frac{x}{3}$.

Thus, $z = \sqrt{x^2 + (2x/3)^2} = \sqrt{13} \cdot \frac{x}{3}$.

Area of the parallelogram is $= 6z = 6 \cdot \sqrt{13} \cdot \frac{x}{3} = 2\sqrt{13}x$.

$$\begin{aligned} 2\sqrt{13}x &= \frac{x^2}{3} \\ x &= 6\sqrt{13} \end{aligned}$$

No partial credit.

Part II

Test 2

Problem II.1 *Find the area of the smallest possible square inscribed in a larger square with a 6-inch long side.*

Solution:

Let the inscribed square be $s \times s$. Let A and B be the endpoints of one of its sides. A and B are on two adjacent sides of the 6×6 square. Let C be the joint vertex of those two adjacent sides. The triangle ABC is right and its legs are $AC = x$ and $CB = 6 - x$ inches long, while the length of the hypotenuse is s inches. Therefore, the area A of the small square is

$$A = s^2 = x^2 + (6 - x)^2,$$

$$A = x^2 + 36 - 12x + x^2,$$

$$A = 2x^2 - 12x + 36.$$

The minimum of this quadratic function is at the vertex, whose x coordinate is $x = \frac{12}{4} = 3$. Thus, $A = 3^2 + (6 - 3)^2 = 18$ in^2 is the minimal area of the inscribed square.

Other simpler methods (such as using the mid point of each leg) that yields the correct answer is acceptable.

Problem II.2 *What is the value of x when you solve the following system of linear equations?*

$$\begin{aligned} 7x - 2y &= -1 \qquad (1) \\ -5x + 3y &= 3 \qquad (2) \end{aligned}$$

Solution:

$$\begin{aligned} (1) \cdot 3 &\implies 21x - 6y = -3 \\ (2) \cdot 2 &\implies -10x + 6y = 6 \\ \text{Add} &: \quad 11x = 3 \\ x &= \frac{3}{11}. \end{aligned}$$

Problem II.3 *Find the solution to the following logarithmic equation.*

$$\log(-x) + \log(x + 2\sqrt{3}) = \log(3)$$

Solution:

$$\begin{aligned}
\log(-x^2 - 2\sqrt{3}x) &= \log(3) \\
-x^2 - 2\sqrt{3}x &= 3 \\
x^2 + 2\sqrt{3} + 3 &= 0 \\
(x + \sqrt{3})(x + \sqrt{3}) &= 0 \\
x &= -\sqrt{3}. \\
& \text{Solution checks.}
\end{aligned}$$

Problem II.4 *Find the exact value of the expression* $\sqrt{17+\sqrt{273}}-\sqrt{17-\sqrt{273}}$.

Solution:

$$\begin{aligned}
\text{Let } n &= \sqrt{17+\sqrt{273}}-\sqrt{17-\sqrt{273}} \\
n^2 &= \left(\sqrt{17+\sqrt{273}}-\sqrt{17-\sqrt{273}}\right)^2 \\
&= 17+\sqrt{273}-2[\sqrt{17+\sqrt{273}}][\sqrt{17-\sqrt{273}}]+17-\sqrt{273} \\
&= 34-2(\sqrt{17^2-273}) = 34-2(\sqrt{16}) = 26 \\
&\therefore \; n = \sqrt{26}
\end{aligned}$$

Problem II.5 *Find the sum* $5 + 6 + \cdots + 194 + 195$.

Solution:

The number of terms is: 191

$$\begin{aligned} & S = 5 + 6 + \cdots + 194 + 195 \\ = \quad & 200 \cdot \frac{191}{2} = 19100. \end{aligned}$$

Problem II.6 *Find the solution to the trigonometric equation* $\cos(2x) = \frac{\sqrt{2}}{2}$ *such that* $\pi < x < \frac{3\pi}{2}$ *(Quadrant III).*

Solution:

$$\begin{aligned}
\cos(2x) &= \frac{\sqrt{2}}{2} \\
2x &= \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4} \\
x &= \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8} \\
\text{In Quadrant III} &: \quad x = \frac{9\pi}{8}.
\end{aligned}$$

Problem II.7 *The equation* $x^2 - 4x + y^2 + 6y = 0$ *represents a circle. Find its radius* r.

Solution:

$$\begin{aligned} x^2 - 4x + y^2 + 6y &= 0 \\ (x-2)^2 + (y+3)^2 &= 4 + 9 = 13 \\ r &= \sqrt{13}. \end{aligned}$$

Problem II.8 *Find the equation of the perpendicular bisector of the line segment AB with endpoints $A(2,-3)$ and $B(4,1)$. (The perpendicular bisector of a line segment is a straight line that passes through the midpoint of the segment and is perpendicular to it.)*

Solution:

The midpoint M has coordinates $\frac{2+4}{2} = 3$ and $\frac{-3+1}{2} = -1$, $M(3,-1)$. The slope of the line segment AB is $\frac{1+3}{4-2} = 2$, thus the slope of the perpendicular line is $-\frac{1}{2}$. The perpendicular bisector is

$$y + 1 = -\frac{1}{2}(x-3)$$

$$y = -\frac{1}{2}x + \frac{1}{2}.$$

Problem II.9 *If the point (3, 6561) is on the graph of* $f(x) = x^n$*, what is* n *?*

Solution:

$$\begin{aligned} 3^n &= 6561 \\ 3^n &= 3^8 \\ n &= 8. \end{aligned}$$

Problem II.10 *Find the angle (in degrees) between the hour hand and the minute hand of a 12-hour clock if the time reads 4:25.*

Solution:
At 1:00 the angle is

$$\frac{360^\circ}{12} = 30^\circ.$$

At 4:25 the angle is

$$0^\circ + \frac{30}{60} \cdot 35 = 17.5^\circ.$$