2010 Stark County High School Mathematics Challenge ANSWER KEY

May 01, 2010

INSTRUCTIONS

You will have **One Hour** to answer each of the following ten questions (in Part I) to the best of your ability. Each question is worth ten points. Please show all of your work. Partial credit will be awarded for sound reasoning and partial solutions. There will be no talking during the test. If you have any questions as to the wording of a problem, please raise your hand and the proctor will assist you. No other questions will be answered.

All tests will be collected promptly at the end of the hour. If you finish early, please close your test booklet and quietly exit the room. No one will be re-admitted to the testing room once they have left; i.e. no bathroom breaks will be allowed. Anyone that is deemed by the proctor to be talking, or otherwise cheating will be asked to leave immediately and will receive a score of zero for the entire competition. You have a short snack break after Part I is completed. Part II will be similar in format, but single answers need to be put in the answer card provided. You are expected to stay for the entire 30 minutes during Part II of the test.

Good Luck, and remember to do your best.

Part I

Test 1

Problem I.1 *How many factors does the number 2010 have?*

Solution

$$\begin{aligned} 2010 &= 2 \cdot 3 \cdot 5 \cdot 67 \\ \#\text{ of factors} &= (1+1)(1+1)(1+1)(1+1) = 16. \end{aligned}$$

They may also list all the factors $\{1, 2, 3, 5, 6, 10, 15, 30, 67, 134, 201, 335, 402, 670, 1005, 2010\}$.

Full credit if they say 16 factors and not list them. Listing all 16 gets full credit. If they list 12 or more, but not all 16, give them +5 points.

Problem I.2 *Prove that for $n = 2, 3, 4, \ldots$*

$$1 < \sqrt[n]{n+1} < 2,$$

where $\sqrt[n]{*}$ refers to the nth root.

Solution

Since $1 < n + 1$, it follows that $\sqrt[n]{1} < \sqrt[n]{n+1}$, which is the first inequality. It remains to prove that $n + 1 < 2^n$. This is true for $n = 2$. In the induction step, use $n + 2 = (n + 1) + 1 < 2^n + 1$ and $2^n + 1 < 2^{n+1}$. The last inequality follows from $1 + \frac{1}{2^n} < 2$.

Partial credit for showing one side of the inequality correctly.

Problem I.3 *Find the sum*

$$\cos(0°)+\cos(1°)+\cos(2°)+\cos(3°)+\cdots+\cos(176°)+\cos(177°)+\cos(178°)+\cos(179°).$$

Solution

Observe that $\cos(180-x) = \cos 180 \cos x + \sin 180 \sin x = (-1)\cos x + (0)\sin x = -\cos x$.

Therefore, $\cos(180-x)+\cos(x) = 0$. Thus,

$$\begin{aligned} & \cos(0°)+\cos(1°)+\cos(2°)+\cos(3°)+\cdots+\cos(176°)+\cos(177°)+\cos(178°)+\cos(179°) \\ = \; & \cos(0°)+[\cos(1°)+\cos(179°)]+[\cos(2°)+\cos(178°)]+\cdots+[\cos(89°)+\cos(91°)]+\cos(90°) \\ = \; & 1. \end{aligned}$$

NPC

Problem I.4 *Let n be a positive integer.*

(a). *Show that*

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

(b). *Use (a) to prove that*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

Solution

Part (a) is trivial. In part (b), let S denote the sum on the left side. Express each term in S using the formula in (a). We have

$$\begin{aligned} S &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \\ &\quad \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right). \end{aligned}$$

All intermediate terms cancel out leaving $S = 1 - \frac{1}{n+1}$.

+5 points for each correct part.

Problem I.5 *The sum of all of the interior angles of* ***seven*** *polygons is* 17×180*. Find the* ***total number of sides*** *of the seven polygons.*

Solution

Let $n_1, n_2, \cdots, n_7$, be the number of sides of the seven polygons. Then, each one of them has a sum of interior angles given by the formula $(n-2) \times 180$. Thus,

$$\begin{aligned}
(n_1 - 2) \times 180 + \cdots + (n_7 - 2) \times 180 &= 17 \times 180 \\
[n_1 + n_2 + \cdots + n_7 - 14] \times 180 &= 17 \times 180 \\
n_1 + n_2 + \cdots + n_7 &= 17 + 14 = 31.
\end{aligned}$$

Consider partial credit for any sound argument that leads to the solution.

Problem I.6 *Given a drawer with 8 blue gloves, 12 black gloves, and 6 green gloves, find the maximum number of gloves you need to pull out to guarantee you have a pair of matching gloves. Assume that each glove in the drawer has exactly one matching pair.*

Solution

It is possible to draw 4 blue gloves, 6 black gloves, and 3 gloves, and still not have a matching pair. That is, $13(= 4+6+3)$ picks may not guarantee a matching pair. But the 14th pick will. So the answer is 14.

NPC

Problem I.7 *Let the sides of a triangle measure a, b, and c length units. Prove that*

$$b^2x^2 + (b^2 + c^2 - a^2)x + c^2 > 0$$

for any real number x. [HINT: Consider the quadratic trinomial in variable x given by $y = b^2x^2 + (b^2 + c^2 - a^2)x + c^2$]

Solution

Let y denote the given quadratic trinomial. Its graph is a parabola opening up because $b^2 > 0$. The discriminant of y is

$$\begin{aligned} D &= (b^2 + c^2 - a^2)^2 - 4b^2c^2 = (b^2 + c^2 - a^2)^2 - (2bc)^2 \\ &= (b^2 + c^2 - a^2 - 2bc)(b^2 + c^2 - a^2 + 2bc) \\ &= [(b - c)^2 - a^2][(b + c)^2 - a^2] \\ &= (b - c - a)(b - c + a)(b + c - a)(b + c + a). \end{aligned}$$

In any triangle, the sum of two side lengths is greater than the length of the third side and this is why $D < 0$. This means that the graph of y has no x-intercepts. Therefore, $y > 0$.

Give some partial credit for using the discriminant and attempting to piece together the necessary factors.

Problem I.8 *If the angles α, β, γ in a triangle satisfy*

$$\sin^2 \gamma = \sin^2 \alpha + \sin^2 \beta,$$

prove that the triangle is a right triangle.

Solution

Method 1:

Use the Law of Sine. Let a, b, c be the lengths corresponding to the sides opposite the angles α, β, γ. Then,

$$\begin{aligned} \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} \\ \sin \alpha &= \frac{a \sin \gamma}{c} \quad \text{and} \quad \sin \beta = \frac{b \sin \gamma}{c}. \end{aligned}$$

Substitute these in $\sin^2 \gamma = \sin^2 \alpha + \sin^2 \beta$.

$$\begin{aligned} \sin^2 \gamma &= \frac{a^2 \sin^2 \gamma}{c^2} + \frac{b^2 \sin^2 \gamma}{c^2} \\ &\therefore\ c^2 = a^2 + b^2 \quad (\text{since we can cancel } \sin^2 \gamma \neq 0) \end{aligned}$$

Thus, the triangle is a right triangle.

Method 2:

Since $\gamma = 180^\circ - (\alpha + \beta)$, it follows that

$$\begin{aligned} \sin \gamma &= \sin(180^\circ - (\alpha + \beta)) \\ &= \sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha. \end{aligned}$$

Therefore,

$$\sin^2 \gamma = \sin^2 \alpha \cos^2 \beta + 2 \sin \alpha \sin \beta \cos \alpha \cos \beta + \sin^2 \beta \cos^2 \alpha.$$

Using the assumption, we get

$$\begin{aligned} 0 &= \sin^2 \alpha (\cos^2 \beta - 1) + 2 \sin \alpha \sin \beta \cos \alpha \cos \beta \\ &+ \sin^2 \beta (\cos^2 \alpha - 1) \end{aligned}$$

and

$$0 = -2 \sin^2 \alpha \sin^2 \beta + 2 \sin \alpha \sin \beta \cos \alpha \cos \beta.$$

When the last equation is divided by $2 \sin \alpha \sin \beta$ (which is positive), it follows that

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta = 0, \text{ that is, } \cos(\alpha + \beta) = 0.$$

This is only possible if $\alpha + \beta = 90^\circ$, which means that $\gamma = 90^\circ$ and the triangle is right.

NPC. Other correct methods get full credit.

Problem I.9 *Simplify the following number and write it in radical form as* $x\sqrt{y}$, *where* $\sqrt{y}$ *is in its simplest form. Here* 9! *denotes the factorial of* 9 *given by* $9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

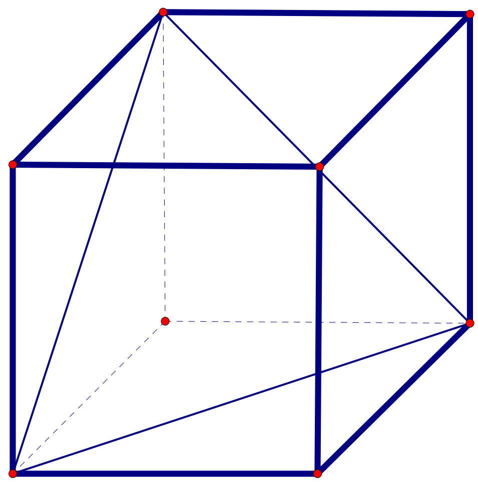
$$\sqrt{9! \cdot 9} + \sqrt{\frac{9!}{9}}$$

Solution

$$\begin{aligned} & \sqrt{9! \cdot 9} + \sqrt{\frac{9!}{9}} \\ = \; & \sqrt{8! \cdot 9^2} + \sqrt{8!} \\ = \; & \sqrt{8!}\,[9 + 1] \\ = \; & 10\sqrt{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ = \; & 80\sqrt{7 \cdot 6 \cdot 5 \cdot 3} = 240\sqrt{70}. \end{aligned}$$

Give +5 points if the answer is in the correct form but not simplified enough.

Problem I.10 *How many **equilateral triangles** can be formed by the vertices of a cube? Pictorial answers will be given credit as long as the answer is correct.*



Solution

Only the diagonals of each square can be the length of a side of an equilateral triangle. We have 12 such diagonals. Each of these diagonals will be the leg of two different equilateral triangles. Since three diagonals are needed to make a triangle, the total number of equilateral triangles is $\frac{1}{3} \cdot 2 \cdot 12 = 8$.

Pictorial answers with the correct count get full credit.

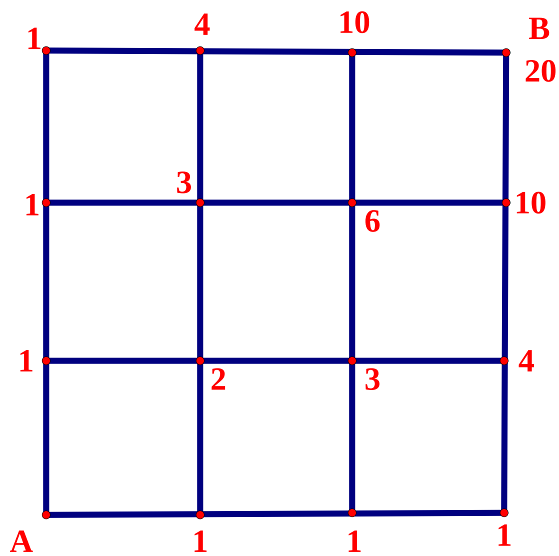
Part II

Test 2

Problem II.1 *Suppose you want to get from point A to point B. However, you are allowed to make one unit move on the grid in either* $\longrightarrow$ *(right direction) or* $\uparrow$ *(up direction) at a time. How many* ***total ways*** *are there to get from point A to point B using such moves only. For example, 3 units* $\longrightarrow$ *followed by 3 units* $\uparrow$ *is considered one legal path from A to B.*

Solution

There are 20 ways. See picture for the number of paths to each node.



Problem II.2 *If* $x > 0$, *find* x *that satisfies*

$$x^{\log_2(x)} = 16,$$

where $\log_2(x)$ *denotes the logarithm of* x *to base* 2.

Solution

$$\begin{aligned}
\log_2(x^{\log_2(x)}) &= \log_2 16 \\
\log_2(x) \cdot \log_2(x) &= 4 \\
(\log_2(x))^2 &= 4 \\
\log_2(x) &= 2 \quad \text{or} \quad \log_2(x) = -2 \\
x &= 4 \quad \text{or} \quad x = \frac{1}{4}.
\end{aligned}$$

+7 points for one answer.

Problem II.3 *Solve the following system of linear equations and find the value of* $\mathbf{x}^{1/2}+\mathbf{y}^{1/3}$.

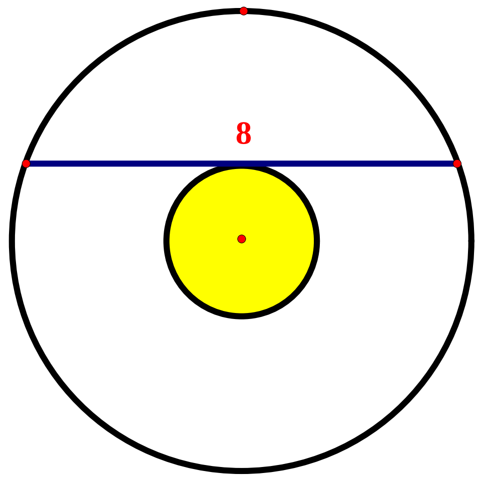
$$\begin{aligned} 3x+2y &= 10 \\ 2x+5y &= 3. \end{aligned}$$

Solution
By method of eliminations or method of substitution, the solutions are

$$\begin{aligned} x &= 4, \quad y=-1 \\ \therefore\ & x^{1/2}+y^{1/3} = \sqrt{4}+\sqrt[3]{-1} = 2-1 = 1. \end{aligned}$$

+7 points for correct x, y *values.*

Problem II.4 *A chord of a circle is tangent to a smaller, concentric circle. Given that the length of the chord is 8, find the area of the donut shape ("annulus") in between the two circles.*



Solution

<u>Method 1</u>: Explore the degree of freedom in the problem and make the small circle radius zero. Then the large circle has radius 4 units. Therefore, the annulus will have area $\pi(4^2) = 16\pi$.

<u>Method 2</u>: Let R and r be the radii of the large and small circles respectively. By Pythagorean Theorem, $R^2 - r^2 = 4^2 = 16$.

$$\begin{aligned}\text{Area of annulus} &= \pi(R^2 - r^2)\\ &= 16\pi.\end{aligned}$$

Problem II.5 *Two positive real numbers have an average of* 10*. Which of the following must be true about the number* μ*, where* μ *is the average of the reciprocals of the original numbers? Put the correct choice on your answer card.*

(a). $\mu = 10$

(b). $\mu = \frac{1}{10}$

(c). μ is any real number

(d). $\mu \geq \frac{1}{10}$

(e). $\mu \leq \frac{1}{10}$

Solution

Let the positive numbers be x, y. Then

$$\begin{aligned} \mu &= \frac{\frac{1}{x} + \frac{1}{y}}{2} \\ &= \frac{\left(\frac{x+y}{2}\right)}{xy} \\ &= \frac{10}{xy}. \end{aligned}$$

If the average of x, y is 10 then the maximum of xy is 100. Therefore, $\mu \geq \frac{10}{100} = \frac{1}{10}$.

Answer is (d).