2011 Stark County
High School
Mathematics Challenge

ANSWER KEY

April 16, 2011

INSTRUCTIONS

You will have **One Hour and Ten minutes** to answer each of the following twelve questions to the best of your ability. Each question is worth ten points. Please show all of your work. Partial credit will be awarded for sound reasoning and partial solutions. There will be no talking during the test. If you have any questions as to the wording of a problem, please raise your hand and the proctor will assist you. No other questions will be answered.

All tests will be collected promptly at the end of the allotted time. If you finish early, please close your test booklet and quietly exit the room. No one will be re-admitted to the testing room once they have left; i.e. no bathroom breaks will be allowed. Anyone that is deemed by the proctor to be talking, or otherwise cheating will be asked to leave immediately and will receive a score of zero for the entire competition. You have a short snack break after the test is completed. There will be a short mathematics presentation following the break.

Good Luck, and remember to do your best.

Problem 1 *Find the largest integer less than 2011 that is divisible by 2,3,4,5,6, and 7 ?*

Solution

$$2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 1680$$

Most will simply multiply and keep dividing by 2. But you must leave out 3.

No partial credit.

Problem 2 *If a student gets 56 on the next test, her final average will be 86. If instead, the student gets a 98 on the next test, her final average will be 92. How many tests has the student taken during the semester (including the last one)?*

Solution

Let x be the total points accumulated by the student before the last test. Let n be the number of tests (including the last one). Then,

$$\begin{aligned} \frac{x+56}{n} &= 86 \\ x &= 86n - 56 \qquad (1) \end{aligned}$$

and

$$\begin{aligned} \frac{x+98}{n} &= 92 \\ x &= 92n - 98 \qquad (2) \end{aligned}$$

Solving (1) and (2),

$$\begin{aligned} 86n - 56 &= 92n - 98 \\ n &= 7 \end{aligned}$$

No partial credit.

Problem 3 *Prove that the* ***sum of the cubes*** *of three consecutive positive integers is divisible by 3. [NOTE: A positive integer a is divisible by a positive integer b, if $a = bc$, for some positive integer c]*

Solution

Let $n-1, n, n+1$ be three consecutive positive integers. Then,

$$\begin{aligned} & (n-1)^3 + n^3 + (n+1)^3 \\ = \; & (n^3 - 3n^2 + 3n - 1) + n^3 + (n^3 + 3n^2 + 3n + 1) \\ = \; & 3n^3 + 6n \\ = \; & 3(n^3 + 2n). \end{aligned}$$

Since $(n^3 + 2n)$ is a positive integer, $(n-1)^3 + n^3 + (n+1)^3$ is divisible by 3.

Full credit for any other correct method.

Problem 4 *At a party 120 handshakes took place. Each person shook hands exactly once with each of the other persons. How many people were at the party?*

Solution 1

Guess and check. Students could jump to the answer once the pattern is discovered. Answer is 16.

# of people	# of handshakes
2	$1 = \frac{1\cdot 2}{2}$
3	$2 = \frac{2\cdot 3}{2}$
4	$6 = 3+2+1 = \frac{3\cdot 4}{2}$
5	$10 = 4+3+2+1 = \frac{4\cdot 5}{2}$
6	$15 = 5+4+3+2+1 = \frac{5\cdot 6}{2}$
7	$21 = 6+5+4+3+2+1 = \frac{6\cdot 7}{2}$
8	$28 = 7+6+5+4+3+2+1 = \frac{7\cdot 8}{2}$
9	$36 = 8+7+6+5+4+3+2+1 = \frac{8\cdot 9}{2}$
10	$45 = 9+8+7+6+5+4+3+2+1 = \frac{9\cdot 10}{2}$
11	$55 = 10+9+8+7+6+5+4+3+2+1 = \frac{10\cdot 11}{2}$
12	$66 = 11+10+9+8+7+6+5+4+3+2+1 = \frac{11\cdot 12}{2}$
13	$78 = 12+11+10+9+8+7+6+5+4+3+2+1 = \frac{12\cdot 13}{2}$
14	$91 = 13+12+11+10+9+8+7+6+5+4+3+2+1 = \frac{13\cdot 14}{2}$
15	$105 = 14+13+12+11+10+9+8+7+6+5+4+3+2+1 = \frac{14\cdot 15}{2}$
16	$\mathbf{120} = 15+13+12+11+10+9+8+7+6+5+4+3+2+1 = \frac{15\cdot 16}{2}$

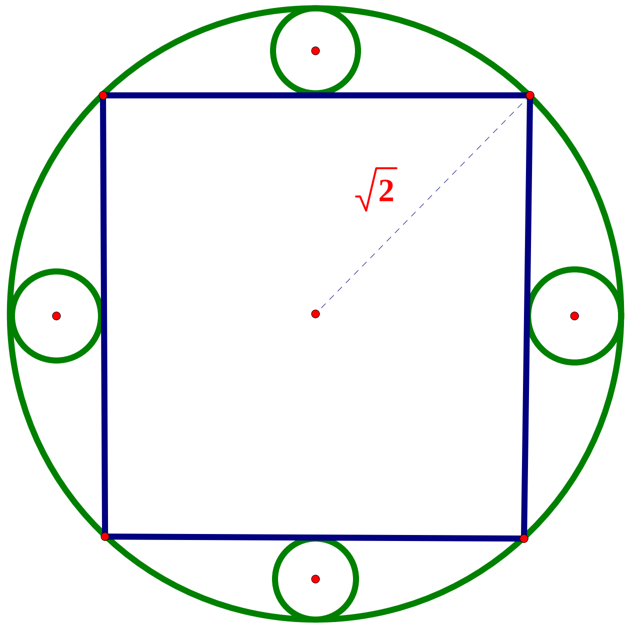
Solution 2

Let n be the number of people. Solve,

$$\begin{aligned} nC2 &= 120 \\ \frac{n!}{(n-2)!2!} &= 120 \\ \frac{(n-1)n}{2} &= 120 \\ n^2 - n - 240 &= 0 \\ (n-16)(n+15) &= 0 \\ n &= 16. \end{aligned}$$

Full credit for any other correct method.

Problem 5 *Find the* ***shaded area*** *in the following figure. The radius of the larger circle is* $\sqrt{2}$ *units. Assume that the quadrilateral inside the circle is a square, and outside this square four congruent smaller circles have been removed. The smaller circles are tangent to both the square and the large circle.*



Solution

Let x be a side of the square. Then, using Pythagorean Theorem,

$$\begin{aligned} x^2 + x^2 &= (2\sqrt{2})^2 \\ 2x^2 &= 8 \\ x^4 &= 4 \Rightarrow x = 2. \end{aligned}$$

Therefore, the radius of each of the smaller circles is $r = \frac{(\sqrt{2}-1)}{2}$. Thus the shaded area is:

$$\begin{aligned} A &= \text{area of big circle - area of square - 4(area of small circle)} \\ &= \pi(\sqrt{2})^2 - 4 - 4\pi\left(\frac{\sqrt{2}-1}{2}\right)^2 \\ &= 2\pi - 4 - 4\cdot\frac{\pi(\sqrt{2}-1)^2}{4} \\ &= 2\pi - 4 - \pi(2 - 2\sqrt{2} + 1) \\ &= (2\sqrt{2}-1)\pi - 4 \simeq 1.7442 \end{aligned}$$

No partial credit.

Problem 6 *Let x be a real number such that $x \neq 0$ and $x > -1$. Use Mathematical Induction to prove that $(1+x)^n > 1 + nx$ for all integers $n \geq 2$.*

Solution

The proof is by induction. The inequality is true for $n = 2$ because $(1+x)^2 = 1 + 2x + x^2 > 1 + 2x$. Then, using the inductive assumption and the fact that $1 + x > 0$, we get

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x) > (1+kx)(1+x) \\ &= 1 + x + kx + kx^2 > 1 + (k+1)x.\end{aligned}$$

No partial credit.

Problem 7 *There are two containers. Container #1 has 8 blue marbles and 12 red marbles. Container #2 has 5 blue marbles and 5 red marbles, You first pick a container at random. Then from that container, you pick one marble at random. What is the probability that the marble you pick is blue?*

Solution
Draw a tree diagram and mark the probabilities along the branches.
Then,

$$\begin{aligned} P(blue) &= \frac{1}{2} \cdot \frac{8}{20} + \frac{1}{2} \cdot \frac{5}{10} \\ &= \frac{1}{5} + \frac{1}{4} = \frac{9}{20} = 0.45 \end{aligned}$$

No partial credit.

Problem 8 *Find all ordered-pairs of real numbers (a, b) that satisfy the system of equations given below.*

$$\begin{aligned} a^2 - ab + b^2 &= 1 \qquad (3) \\ 3a^2 + 2ab - 2b^2 &= 3 \qquad (4) \end{aligned}$$

Solution

$$\begin{aligned} 2 \times (3) &\Rightarrow 2a^2 - 2ab + 2b^2 = 2 \\ (4) &\Rightarrow 3a^2 + 2ab - 2b^2 = 3 \\ \text{add} &: 5a^2 = 5 \\ a &= \pm 1 \end{aligned}$$

If $a = 1 \overset{\text{eq. (3)}}{\Rightarrow} 1 - b + b^2 = 1 \Rightarrow b(b-1) = 0 \Rightarrow b = 0$ or 1.
If $a = -1 \overset{\text{eq. (3)}}{\Rightarrow} 1 + b + b^2 = 1 \Rightarrow b(b+1) = 0 \Rightarrow b = 0$ or -1.
Therefore, the solution set is $\{(1,0), (1,1), (-1,0), (-1,-1)\}$.

Two correct pairs gets 5 points.

Problem 9 *Find all solutions to the following trigonometric equation on the interval* $[0, 2\pi)$.

$$3 + 3\sin x = 2\cos^2 x$$

Solution

$$\begin{aligned}
3 + 3\sin x &= 2\cos^2 x \\
3 + 3\sin x &= 2(1 - \sin^2 x) \\
2\sin^2 x + 3\sin x + 1 &= 0 \\
(2\sin x + 1)(\sin x + 1) &= 0 \\
\sin x &= -\frac{1}{2} \text{ or } \sin x = -1 \\
x &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{3\pi}{2}.
\end{aligned}$$

No partial credit.

Problem 10 *Find the equations of the tangent lines which touch the circle $x^2 + (y+1)^2 = 10$ at the points with the x-coordinate equal to 1. [HINT: There are two of them. A picture may help.]*

Solution

If $x = 1$, the y-coordinate of the point on the given circle is found from the equation $(y+1)^2 = 9$.

Therefore, there are two such points: $A(1,2)$ and $B(1,-4)$.

The slope of the line going through the center of the circle $C(0,-1)$ and A is 3, so the slope of the tangent line at A is $-\frac{1}{3}$. The equation of this tangent line is:

$$\begin{aligned} y - 2 &= \frac{-1}{3}(x-1) \\ y &= -\frac{1}{3}x + \frac{7}{3}. \end{aligned}$$

The slope of the line going through the center of the circle $C(0,-1)$ and B is -3, so the slope of the tangent line at A is $\frac{1}{3}$. The equation of this tangent line is:

$$\begin{aligned} y - (-4) &= \frac{1}{3}(x-1) \\ y &= \frac{1}{3}x - \frac{13}{3}. \end{aligned}$$

Give 5 points for one answer.

Problem 11 *Let a and b be real numbers.*

(a). Find an example showing that $(a^b)^2 \neq a^{(b^2)}$. (This means that the operation of exponentiation is not associative.)

(b). However, $(a^b)^2 = a^{(b^2)}$ is satisfied for some a and b. Find *all* such numbers.

Solution
(a) One example is $a = 2$ and $b = 3$: $(2^3)^2 = 8^2 = 64$, whereas $2^{(3^2)} = 2^9 = 512$.

(b) If both powers are defined, the equation $(a^b)^2 = a^{(b^2)}$ is equivalent to $a^{2b} = a^{(b^2)}$. This is immediately true if $a = 0$ and b is positive, as well as if $a = 1$ and b is any real number. Otherwise, $2b = b^2$ would have to hold true. This gives $b = 0$ or $b = 2$. Both powers are indeed defined in all these cases. All solutions are therefore the following pairs (a, b):
$(0, b)$ for any positive real number b,
$(1, b)$ for any real number b, and
$(a, 0)$ and $(a, 2)$ for any real number a.

Give 2.5 points for part (a) and 2.5 points for each of the answers in part (b).

Problem 12 *Find the solution(s) of* $\log_2(x) + \log_4(27x) = 3$. *Assume* $x > 0$.

Solution

$$\begin{aligned}
\log_2(x) + \log_4(27x) &= 3 \\
\log_2(x) + \frac{\log_2(27x)}{\log_2 4} &= 3 \\
\log_2(x) + \frac{1}{2}\log_2(27x) &= 3 \\
\log_2\left[x\sqrt{27x}\right] &= 3 \\
x\sqrt{27x} &= 2^3 \\
27x^3 &= 64 \\
x^3 &= \frac{64}{27} \\
x &= \frac{4}{3}.
\end{aligned}$$

No partial credit.