Lecture 14—Dimensional Analysis

First, let's talk about dimensions. Every physical property we work with has dimensions—you may know these as the "units" on something. Take, for example, pressure. There's a *lot* of ways we could describe the units on pressure. It could be in psi. It could be in psf. It could be N/m^2. It can even be a Pascal. In the end, though, all these definitions have one thing in common. They are *all* a force divided by an area. So, the *dimensions* on pressure are FL^{-2}. See how nice this is? Ok, let's try another one. Kinematic viscosity. Look on your sheet—it's often in square meters per second. That'd be L^2T^{-1}. Easy, right? There's only one wrinkle. There are two common systems of dimension. One uses force, length, and time. The other uses mass, length, and time (these are abbreviated FLT and MLT). Since a force is just a mass times an acceleration, you can *convert* between one and the other, but notice that pressure becomes MLT^{-2} in the MLT system (and notice that density goes from a relatively nice ML^{-3} to $FL^{-4}T^2$). Either system works—just go with whichever makes the most sense for your problem. Oh, and you *sometimes* need a fourth dimension—temperature.

Let's take a familiar example for how to perform a dimensional analysis—the Darcy friction factor for pipes. We know that friction factor (*f*) is a function of….the average velocity (*V*), the pipe diameter (*D*), the fluid density (*ρ*), the viscosity (*μ*) and the pipe roughness (*ε*).

$$f = f(V, D, \rho, \mu, \varepsilon)$$

This looks daunting at first (especially if you're an experimental scientist). Picture what would have to happen to make an experimental matrix to determine friction factor! We'd need to hold all the other variables constant and then vary just one variable, then repeat. In short, we've got a *huge* experimental matrix and we'll be here a while. *THIS* is what dimensional analysis is good for. We can express this problem (and others like it) in the following form:

$$\alpha = C_\pi a_1^{\,c1} a_2^{\,c2} a_3^{\,c3} \cdots a_n^{\,cn}$$

Where C_π is a dimensionless coefficient. This looks bad, but all it says is that some dependent variable *α* is the product of some constant times all the independent variables raised to constant powers. We know that the dimensions of *α* must equal the dimensions of the right hand side of the equation, so the fun begins. Consider the friction factor example:

$$f = C_\pi V^{c1} D^{c2} \rho^{c3} \mu^{c4} \varepsilon^{c5}$$

The dimensions on each variable are:

$f = 1$
$V = LT^{-1}$
$D = L$
$\rho = FT^2L^{-4} = ML^{-3}$
$\mu = FTL^{-2} = ML^{-1}T^{-1}$
$\varepsilon = L$

Since the left and right sides of the equation must have the same dimensions, we can set up the following:

$$F^0L^0T^0 = \left(LT^{-1}\right)^{c1}\left(L\right)^{c2}\left(FL^{-4}T^2\right)^{c3}\left(FL^{-2}T\right)^{c4}\left(L\right)^{c5}$$

Which we can rewrite to:

$$F^0L^0T^0 = F^{c3+c4}L^{c1+c2-(4c3)-(2c4)+c5}T^{-c1+(2c3)+c4}$$

So,

for F	$c_3 + c_4 = 0$
for L	$c_1 + c_2 - (4c_3) - (2c_4) + c_5 = 0$
for T	$-c_1 + (2c_3) + c_4 = 0$

And,

$c_3 = -c_4$	from the equation for F
$c_1 = 2(-c_4) + c_4 = -c_4$	from the equation for T
$c_2 = c_4 - (4c_4) + (2c_4) - c_5 = -c_4 - c_5$	from the equation for L

Substituting these back into the original equation:

$$f = C_\pi V^{-c4} D^{-c4-c5} \rho^{-c4} \mu^{c4} \varepsilon^{c5}$$

and finally,

$$f = C_\pi \left(\frac{VD\rho}{\mu}\right)^{-c4} \left(\frac{\varepsilon}{D}\right)^{c5} \Rightarrow f = f\left(\frac{VD\rho}{\mu}, \frac{\varepsilon}{D}\right)$$

So we've reduced the number of independent variables from five to two! This is a much nicer experimental matrix. The two dimensionless groups we've created are called *pi terms* (more on this later).

Some important observations:

- There's nothing that says we can't invert, square, take the sine of, or combine pi terms. They're dimensionless and they'll stay that way.
- This analysis would have worked exactly the same way in MLT. Try it!
- Notice that the first pi term is a Reynolds number! Many of our favorite dimensionless numbers fall out of dimensional analysis and are, in fact, pi terms. It makes sense to see them in fluid mechanics problems

Here's some common ones:

Reynolds	$\frac{Vl\rho}{\mu}$	where viscosity is important (laminar flow)
Froude	$\frac{V}{\sqrt{gl}}$	where gravity effects are important
Weber	$\frac{\rho V^2 l}{\sigma}$	where surface tension is important
Euler	$\frac{p}{\rho V^2}$	where static pressure is important (conduits)

- It's no accident we ended up with just two dimensionless groups. There's a theorem to explain how many you get. You take the number of *independent* variables you have and subtract the number of *dimensions* you used. What's left is the number of pi groups you can expect. This truth is called the Buckingham Pi theorem after its inventor.

As a parting thought, suppose we hadn't started with both density and dynamic viscosity in our analysis? We *could* do this whole thing by saying "look, I suspect that density and viscosity can be dealt with fully by combining the two into kinematic viscosity." You could say $v=\mu/\rho$ and be done with it. What would happen? Wait! *Now* there's only *four* independent variables, so doesn't the Buckingham Pi Theorem suggest that we can expect only *one* dimensionless variable? Well, no. Do the dimensions!

$f = 1$
$V = LT^{-1}$
$D = L$
$v = L^2T^{-1}$
$\varepsilon = L$

So there's only two dimensions! So, only four variables, but only two dimensions, so still two groups. Try it yourself, and you'll see you get the same answer again, with one important difference—the math was a whole lot easier. This points out an important aspect of dimensional analysis—you need to choose an appropriate set of variables. A lot of dimensional analysis is trying to winnow down a large variable list into a few variables that make sense. Let's take a look…