Lecture 15—Dimensional Analysis Revisited

Ok, so last time we managed to explain that there's a *reason* the Moody diagram plots friction factor as a function of Reynolds number and relative roughness. Is dimensional analysis really that useful, though?

YES. In addition to providing fewer variables and getting rid of units, dimensional analysis can suggest ways of looking at your data that provide more insight than you might expect.

Here are two examples:

Suppose that we did some experiments on drag force in a wind tunnel. Basically, we stuck spheres of known diameter in the tunnel and threw various wind speeds at them, then determined the force of drag exerted on the sphere. Through the judicious use of HeaviAir™, we were also able to vary the fluid density quite a bit. Lastly, for reasons unknown even to us, we chose to do the whole thing in English units. For the works, $\mu = 4 \times 10^{-7}$ lb·sec/ft^2.

Test	V (fps)	D (ft)	ρ (slug/ft^3)	F_d (lb)
1	0.18	0.1	0.00222	2.82×10^{-7}
2	1.8	0.1	0.00222	1.326×10^{-5}
3	18	0.1	0.00222	0.001326
4	180	0.1	0.00222	0.1326
5	100	1.8×10^{-4}	0.00222	2.82×10^{-7}
6	100	0.0018	0.00222	1.326×10^{-5}
7	100	0.018	0.00222	0.001326
8	100	0.18	0.00222	0.1326
9	100	1.8	0.00222	2.82
10	100	18	0.00222	565
11	100	0.1	4×10^{-6}	1.572×10^{-4}
12	100	0.1	4×10^{-5}	7.38×10^{-4}
13	100	0.1	4×10^{-4}	0.00738
14	100	0.1	0.004	0.0738
15	100	0.1	0.04	0.1572
16	100	0.1	0.4	3.14

We can plot all sorts of plots relating each independent variable to the dependent variable, all of which tell us *something* about how drag force is related to velocity, diameter, and fluid density. However, let's try a little dimensional analysis.

$$F_d = f(V, D, \rho, \mu)$$

$$F_d = C_\pi V^{c1} D^{c2} \rho^{c3} \mu^{c4}$$

$$F^1 L^0 T^0 = \left(LT^{-1}\right)^{c1} L^{c2} \left(FT^2 L^{-4}\right)^{c3} \left(FTL^{-2}\right)^{c4}$$

$$F^1 L^0 T^0 = F^{c3+c4} L^{c1+c2-(4c3)-(2c4)} T^{-c1+(2c3)+c4}$$

and therefore

c3 = 1 – c4
c1 = 2c3 + c4 = 2(1 – c4) + c4 = 2 – c4
c2 = -c1 + 4c3 + 2c4 = -2 + c4 + 4(1 – c4) + 2c4 = -2 + c4 + 4 – 4c4 + 2c4 = 2 – c4

from which we get,

$$F_d = C_\pi V^{2-c4} D^{2-c4} \rho^{1-c4} \mu^{c4}$$

$$F_d = C_\pi \rho V^2 D^2 \left(\frac{VD\rho}{\mu}\right)^{-c4}$$

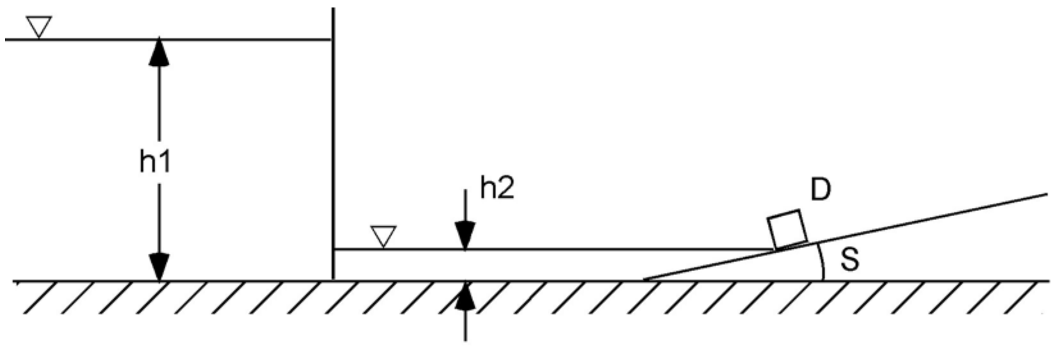
and finally,

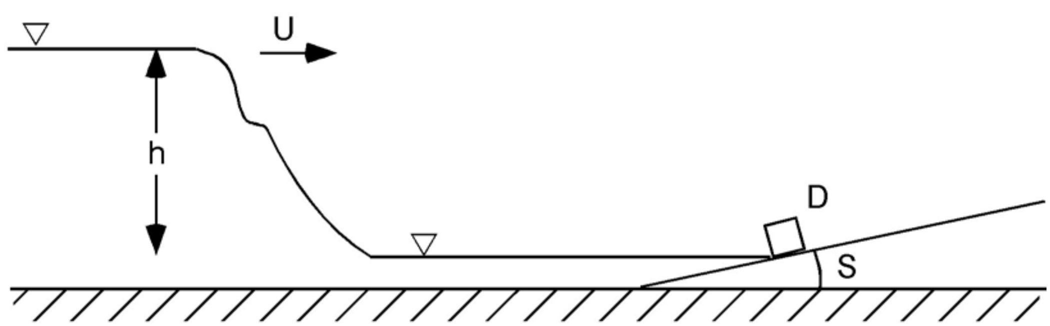
$$\frac{F_d}{\rho V^2 D^2} = f(R)$$

which should look substantially more familiar. Note that this provides, effectively, a second pi group—one we'll call drag coefficient. See? Dimensional analysis can also be used to suggest ways of looking at your data that make more sense than just plots of one independent variable against the dependent variable.

Ok. Remember how I explained last time we spoke about how one of the biggest problems in dimensional analysis is just knowing which variables to choose? This is a big deal. Sometimes we can combine variables together (as in the case with kinematic viscosity), and sometimes we can eliminate a few by making some simplifying assumptions or through some knowledge of fluid mechanics. Here’s a hairy real world example (my dissertation!)

Suppose we have:





The following variables may (or may not) be important to our dependent variable, ℓ.

h1 initial impoundment height
h2 initial water depth
h bore height
U flow velocity
c wave celerity
f friction factor

S	slope
D	particle diameter
CSF	shape factor
ρ	fluid density
ρ_s	sediment density
μ	fluid viscosity
g	gravity
w	settling velocity

How on earth do we winnow this list down? We can start by saying that h is a function of h1 and h2 (this is from something called the "dam-break problem"), so we only need h. We can define $\gamma'=g(\rho_s-\rho)$, which allows us to get rid of any two of the three (and add in γ'); I choose g and ρ_s. w=f(D, CSF, γ'), so I only need w, and can be rid of the others. Next, I know from fluid mechanics that R is really big here. So big that the flow is fully turbulent, and therefore viscosity is unimportant to the system. μ gets voted off the island. Now it's time for my experimental assumptions. I chose to make slope, friction factor, and fluid density constant. This substantially reduces the list. Now ℓ=f(h, U, w). From here it doesn't take a rocket scientist to figure out two obvious pi groups, ℓ/h and w/u. The important thing to note is that *it didn't have to be that way*. I could have chosen to make a Froude number (a pretty obvious thing for bores), $\frac{U}{\sqrt{gh}}$, and a "particle" Froude number $\frac{w}{\sqrt{gl}}$. Plotting *these*, however, doesn't really aid our understanding of the problem any. We don't get any coherent trends, so these pi groups don't really work for us.

Unfortunately, this isn't the only way to approach this problem. Let's try a force ratio instead. The *forces* important to this problem are the drag force on the particle (we hope) and the weight of the particle.

$$F_g = \gamma' D^3 \qquad\qquad F_d = \rho C_D D^2 U^2$$

so we *could* make a force ratio:

$$\frac{2\gamma' D^3}{\rho C_D D^2 U^2} = \frac{2\gamma' D}{\rho C_D U^2} = \frac{w^2}{U^2}$$

Nice! But, we're still stuck trying to normalize ℓ to something. All sorts of length scales come to mind, and only by seeing which ones aid our understanding of the data can we figure out which one to use. This is dimensional analysis at its best (and worst).