Lecture 16—Dimensional Analysis and Settling Velocity

We started this whole thing off by saying we wanted to gain insight into settling velocity using dimensional analysis. Maybe we should get back to that.

Here's a list of independent variables for settling velocity:

D	particle diameter
μ	viscosity
ρ	fluid density
ρ_s	sediment density
g	gravity

Let's do the usual combination of $\gamma' = (\rho_s - \rho)g$ and get rid of g and ρ_s. Now we're left with:

D	L
μ	$ML^{-1}T^{-1}$
ρ	ML^{-3}
γ'	$ML^{-2}T^{-2}$

(Remember that w has units of LT^{-1}, and note that I'm doing this in the MLT system, not FLT. It still works.)

Anyway, we've got five variables and three dimensions, so two pi groups should cover it. Let's do the formalism again, just to set it down.

$$f = f\left(C_\pi D^{c1} \mu^{c2} \rho^{c3} \gamma'^{c4} w^{c5}\right)$$

$$M^0 L^0 T^0 = (L)^{c1} \left(ML^{-1}T^{-1}\right)^{c2} \left(ML^{-3}\right)^{c3} \left(ML^{-2}T^{-2}\right)^{c4} \left(LT^{-1}\right)^{c5}$$

$$M^0 L^0 T^0 = M^{c2+c3+c4} L^{c1-c2-3c3-2c4+c5} T^{-c2-2c4-c5}$$

So,

$$c_2 + 2c_4 + c_5 = 0 \quad \Rightarrow \quad c_5 = -c_2 - 2c_4$$
$$c_2 + c_3 + c_4 = 0 \quad \Rightarrow \quad c_3 = -c_2 - c_4$$
$$c_1 - c_2 - 3c_3 - 2c_4 + c_5 = 0 \quad \Rightarrow \quad c_1 - c_2 + 3c_2 + 3c_4 - 2c_4 - c_2 - 2c_4 = 0 \quad \Rightarrow \quad c_1 = -c_2 + c_4$$

from which,

$$f = C_\pi D^{-c2+c4} \mu^{c2} \rho^{-c2-c4} \gamma'^{c4} w^{-c2-2c4}$$

and

$$f = \left(\frac{\rho D w}{\mu}\right)^{-c2} \left(\frac{\gamma' D}{\rho w^2}\right)^{c4}$$

Well, so tell you what, let's do like before, and make two dimensionless groups:

$$\pi_1 = \frac{wD\rho}{\mu}$$

and

$$\pi_2 = \frac{\gamma' D}{\rho w^2}$$

It turns out that this is not so exciting as we might have hoped. π_1 is just a particle Reynolds number (who knew?), but also:

$$\pi_2 = \frac{\gamma' D}{\rho w^2} \propto \frac{\frac{\pi}{6}\gamma' D^3}{\frac{\pi}{4}\rho w^2 D^2} = C_D$$

So we've really just described C_D vs. R again. Remember, though, that these groups are supposed to be an aid to our understanding of the system, so there's nothing stopping us from grouping the groups. One way of doing this is to take:

$$\pi_1^{\ 2}\pi_2 = \frac{w^2 D^2 \rho^2}{\mu^2} \cdot \frac{\gamma' D}{\rho w^2} = \frac{\rho \gamma' D^3}{\mu^2} = \frac{\gamma' D^3}{\rho \nu^2} = D_*$$

and

$$\frac{\pi_1}{\pi_2} = \frac{wD\rho}{\mu} \cdot \frac{w^2 \rho}{D\gamma'} = \frac{w^3 \rho^2}{\gamma' \mu} = \frac{w^3 \rho}{\gamma' \nu} = w_*$$

Note that the first group doesn't have settling velocity in it, and the second one doesn't have grain diameter in it! This is nice—we could plot D_* against w_* to make plots of settling velocity independent of units! And sure enough, if you look in Dietrich, that's what axes he uses. D_* is often called *dimensionless grain diameter* and w_* *dimensionless settling velocity*, but you now know them to be pi groups that fall naturally out of dimensional analysis of settling velocity. D_* is, on one hand, just grain diameter with appropriate constants added to make the dimensions go away. On the other hand, it's the natural result of multiplying $C_D R^2$. Same goes for w_*. It's just w times some things to make the dimensions go away, or it's R/C_D. Now, go back and read those papers again!