# Lecture 21—Suspended Sediment Transport

We're interested in expressions for q_s, the rate of sediment transport. This is often divided into two segments, bedload and suspended load. As it turns out 50-99% of a river's sediment moves as suspended load. If we wanted to know the *amount* of sediment moving through a reach in a given year (maybe because we were worried about how much sediment was ending up in a reservoir), we'd be very interested in suspended load.

One problem is that we can't just measure the amount of suspended load. Suspended load has a distribution, just like velocity does, so the same amount of sediment isn't being carried throughout the water column. Often, rivers are very deep, so we have to *estimate* the amount of suspended load from a few samples. Now, if we knew the *distribution* of suspended load in the water column, we could determine the sediment transport rate.

So, how to determine the concentration of suspended sediment with depth? Basically, for a steady concentration profile, an equilibrium has been reached between the flux of sediment down (by gravity) and the flux of sediment *up* by diffusion.
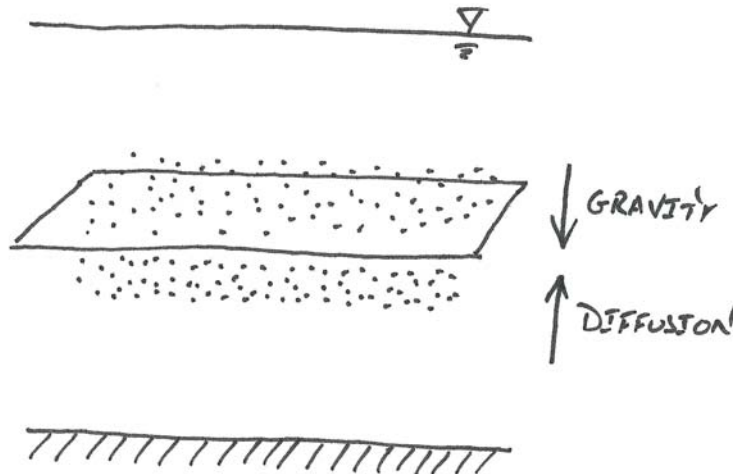
We have an expression for the flux of material down:

$$\text{flux} = w_p c$$

So all we need is an expression for the flux up, and we'll be done! This *seems* easy; what drives the flux up is diffusion (the same force that moves ink evenly throughout a jar of water, for instance). We know of *lots* of examples of diffusive processes—examples include heat flow and groundwater flow. Mathematically, these all look the same. Flux is proportional to gradient, with the coefficient of proportionality being some material property of the medium. Like this:

$$\text{flux} = k\,\frac{\mathrm{d}i}{\mathrm{d}x}$$

where the derivative is the gradient of interest (temperature change for heat, hydraulic head for groundwater) and k is the coefficient of proportionality (specific heat for heat, permeability for groundwater). In *our* case, the gradient is the change in sediment concentration with depth, and the coefficient is basically unknown:

$$\text{flux} = -\varepsilon\,\frac{\mathrm{d}c}{\mathrm{d}y}$$

where ε is some property of the fluid called the *diffusion coefficient* (the minus sign is just to make sure we know the flux is against gravity).

Now we're stuck. We have no means of determining ε without rather lengthy and confusing experiments. However, once again Osbourne Reynolds comes through in a pinch, with the following analogy.

Reynolds reasoned that the diffusion coefficient should act very like another diffusion equation we've dealt with; the diffusion of momentum. We *have* a statement about how momentum is transmitted in a laminar fluid:

$$\tau = \mu\,\frac{\mathrm{d}u}{\mathrm{d}y}$$

and from a long time ago, we have a statement for the way momentum is transmitted in a *turbulent* system (which is what we're really interested in anyway, since turbulence is what's driving this diffusion):

$$\tau = \rho \varepsilon_m \frac{d\bar{u}}{dy}$$

where ε_m is called the *turbulent eddy diffusivity* or *kinematic eddy viscosity* (because it has the form of a kinematic viscosity). This is related to the length scale over which eddies transport momentum (which is in turn affected by the viscosity; you get the drift). Reynolds' brilliant, twisted, mad insight was this; the diffusion coefficient (that says how suspended sediment is transported) is proportional to the eddy viscosity (that says how momentum is transported).

$$\varepsilon = \beta \varepsilon_m$$

where β is a proportionality constant Reynolds claims is about equal to 1.

We're actually worse off than we were before. Instead of one unknown (ε), we're stuck with two (β and ε_m). On the other hand, this allows us all sorts of fun if only we knew how ε_m varied with depth (and if β really acts like Reynolds says it does).

There are lots of choices for how ε_m varies with depth:

Let's think about it, though. We *know* how τ varies in a flow:
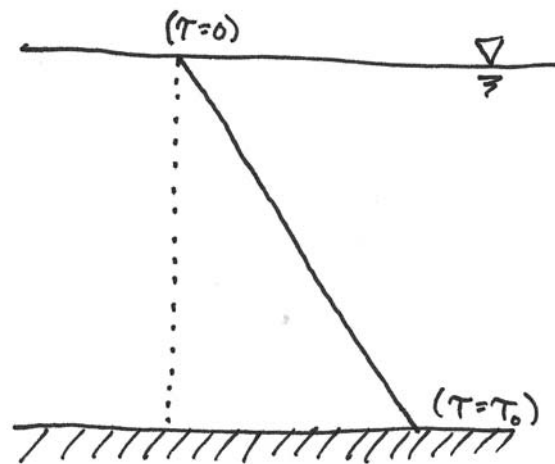
so we have,

$$\tau = \tau_0\left(1 - \frac{y}{d}\right) = \rho u_*^{\,2}\left(1 - \frac{y}{d}\right)$$

and, from the law of the wall:

$$\bar{u} = U + \frac{u_*}{\kappa}\left(1 + \ln\left(\frac{y}{d}\right)\right)$$

so that,

$$\frac{d\bar{u}}{dy} = \frac{u_*}{\kappa y}$$

Combining our two equations for τ,

$$\rho u_*^{\,2}\left(1 - \frac{y}{d}\right) = \rho \varepsilon_m \frac{u_*}{\kappa y}$$

Which yields:

$$\varepsilon_m = \kappa u_* y\left(1 - \frac{y}{d}\right)$$

which is basically a parabola. NOW, if $\varepsilon = \beta\varepsilon_m$, then,

$$\varepsilon = \beta\left(\kappa u_* y\left(1 - \frac{y}{d}\right)\right)$$

so at long last we can make the balance we wanted:

$$w_p c = -\varepsilon\frac{dc}{dy}$$

rearranging,

$$\frac{dc}{dy} + \frac{w_p}{\varepsilon}c = 0$$

and substituting,

$$\frac{dc}{dy} + \frac{w_p}{\beta\kappa u_* y\left(1 - \dfrac{y}{d}\right)}c = 0$$

This is rather unpleasant. Let's simplify. Take:

$$z = \frac{w_p}{\beta\kappa u_*} \qquad\qquad \eta = \frac{y}{d}$$

Our equation simplifies to:

$$\frac{dc}{c} = \frac{-z\,d\eta}{\eta(1 - \eta)}$$

Integrating,

$$\ln c + c_0 = z\ln\left(\frac{1 - \eta}{\eta}\right)$$

And we're back to square one. We *still* have an unknown (c_0), and we can't pick it at $y=0$ by the terms of our derivation for ε_m. We *could*, however, choose it at some arbitrary level, $y=a$. This results in:

$$\frac{c}{c_a} = \left(\frac{d-y}{y} \cdot \frac{a}{d-a}\right)^z$$

which is called the Rouse Equation.

There are several things we can do with this. One is just a clarification; the reason we couldn't pick c_0 at $y=0$ is that ε_m goes to 0 (it also does this at the surface). To solve the *surface* inaccuracies, some people choose:

$$\begin{cases} \dfrac{y}{d} < 0.5 & \varepsilon_m = \kappa u_* y\left(1 - \dfrac{y}{d}\right) \\ \\ \dfrac{y}{d} > 0.5 & \varepsilon_m = \text{constant} \end{cases}$$

The second thing is that z is effectively the slope of $\log(c)$ vs. $\log(\frac{d-y}{y})$. The *neat* thing is that $z\beta = P$, and $\beta \approx 1$. SO, if this all works, we have a means of a) determining concentration profile (by determining the Rouse number of the grain size in question, then getting a single reference value), and b) checking the validity of our assumption that $\beta=1$. Here,

$$\beta = \frac{w_p}{z\kappa u_*}$$

Experimentally, this has turned out to be $0.93 < \beta < 1.10$ for sands and silts.

This still isn't great, since we need a reference concentration to make the whole thing work. As a simplification, though, near the bed:
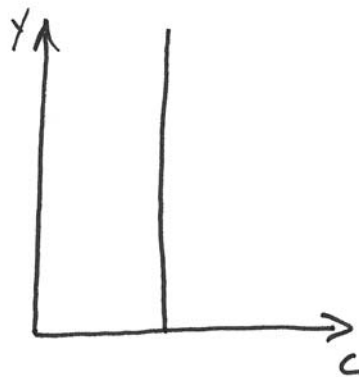
$$\varepsilon = \beta u_* \kappa y \qquad \frac{y}{d} << 1$$

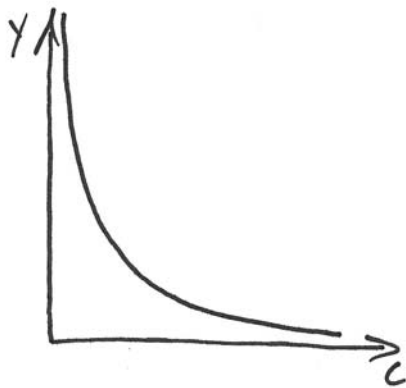This results in a much more solvable equation for *c*, resulting in:

$$\frac{dc}{c} = -z \frac{dy}{y}$$

So, what are some results of this?

If *z*=0, *c*=constant

If *z*=1, $c=y^{-1}$

for *z*>1 bedload dominates quickly

The figure shows relative concentration distribution. The vertical axis is labeled "Value of $\frac{y-a}{d-a}$" ranging from 0 to 1.0, with "Surface" marked at 1.0 and "Bottom" marked at 0. The horizontal axis is labeled "Relative concentration, $\frac{c}{c_a}$" ranging from 0 to 1.0. Curves are labeled with "Value of z" values: 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{32}$. The notation $\frac{a}{d} = 0.05$ is marked near the bottom.