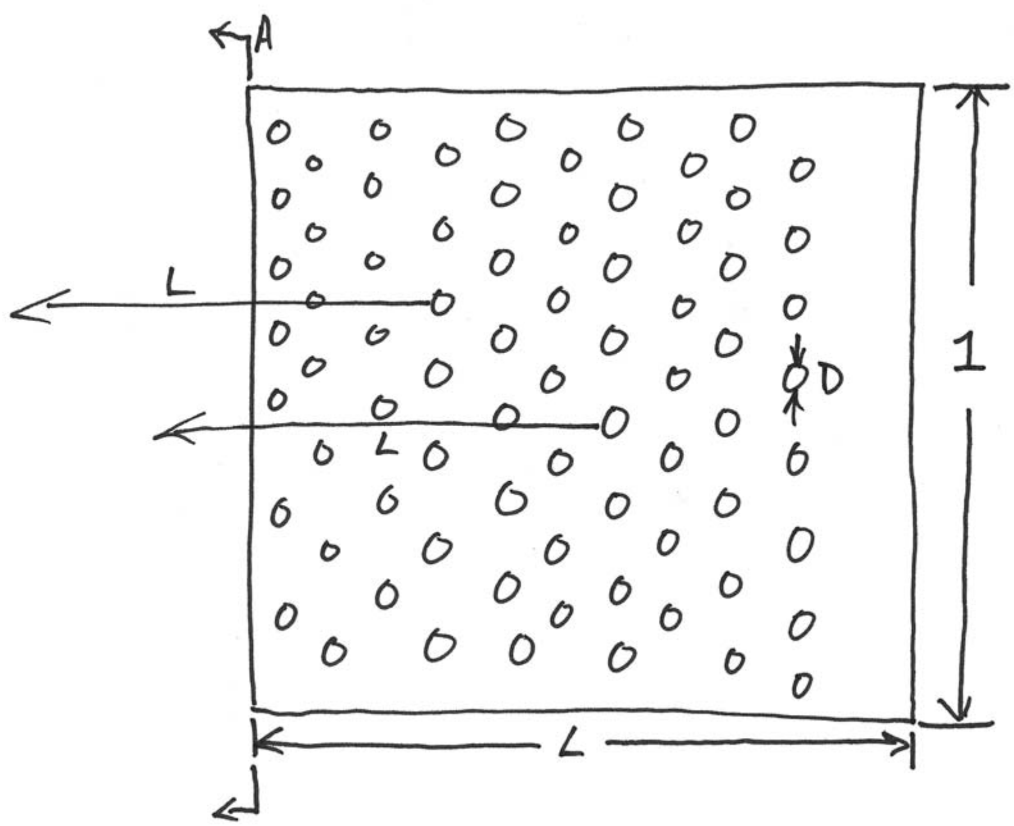
Lecture 24—Probability Bed Load Equations

Albert Einstein claimed that he became a physicist because he found sediment transport (in which he was originally trained) too intractable a problem. How typical, then, that his son decided to one-up dad by taking the problem on.

H.A. Einstein, in 1942, tried a very different tack from the excess shear concept. Einstein described a turbulent fluid where forces vary in both space and time; the *probability* of movement of a particle depends on the *probability* that the fluid forces will exceed resistive forces.

So, let's consider a flat bed:



And let’s assume that if a particle moves, it will move a distance *L*, which we can describe in terms of grain diameters, $L=\lambda_0 D$.

From before,

$$q_b = \frac{g_b}{(\rho_s - \rho)g}$$

And, from Einstein,

$$q_b = \frac{L}{A_1 D^2} \cdot \wp_s \cdot A_2 D^3 = \frac{A_2}{A_1} \lambda_0 D^2 \wp_s$$

so,

$$\frac{g_b}{(\rho_s - \rho)gD^2} \cdot \frac{A_1}{A_2} = \lambda_0 \wp_s$$

We would like $\wp_s$ to be independent of time.

$$\wp = t \wp_s$$

Einstein assumes $t \propto \frac{D}{w_s}$, which is the time it takes for a grain to fall one diameter. From long ago,

$$w_s{}^2 = \frac{4g}{3C_D} D \left(\frac{\rho_s - \rho}{\rho} \right)$$

so

$$\frac{D}{w_s} = \frac{1}{\sqrt{\frac{4}{3}C_D}}\sqrt{\frac{D\rho}{g(\rho_s - \rho)}}$$

and

$$t = \frac{A_3}{F}\sqrt{\frac{D\rho}{g(\rho_s - \rho)}}$$

where $F = \sqrt{\frac{4}{3}C_D}$ for spheres, and is given implicitly by Rubey (1933) to be:

$$F = \sqrt{\frac{2}{3} + \frac{36\mu^2}{gD^3\rho(\rho_s - \rho)}} - \sqrt{\frac{36\mu^2}{gD^3\rho(\rho_s - \rho)}}$$

Now we can formulate $\wp$.

$$\wp = \frac{A_1 A_3}{\lambda_0 A_2}\left[\frac{1}{F}\frac{g_b}{(\rho_s - \rho)g}\sqrt{\frac{\rho}{\rho_s - \rho}}\frac{1}{g^{\frac{1}{2}}D^{\frac{3}{2}}}\right]$$

Additionally, Einstein assumes that the probability will be some function of particle weight over the average lift on the particle. So,

$$\wp = f\left(\frac{A_2 D^3(\rho_s - \rho)g}{A_4 D^2 \rho V^2}\right)$$

Which *V* to choose? *V* is taken as the velocity at the top of the laminar boundary layer, and is expressed as V=11.6u_*. This makes our equation:

$$\wp = f\left(\frac{A_2 D^3(\rho_s - \rho)g}{A_4 D^2(135gRS)}\right)$$

Simplifying,

$$\wp = f\left(\frac{A_2}{135A_4}\cdot\frac{\rho_s - \rho}{\rho}\cdot\frac{D}{RS}\right)$$

Now, let $A = \frac{A_1 A_3}{\lambda_0 A_2}$, $B = \frac{A_2}{135A_4}$. This makes the whole Einstein equation:

$$A\left[\frac{1}{F}\left(\frac{g_b}{(\rho_s - \rho)g}\sqrt{\frac{\rho}{(\rho_s - \rho)}}\frac{1}{g^{\frac{1}{2}}D^{\frac{3}{2}}}\right)\right] = f\left(B\left(\frac{\rho_s - \rho}{\rho}\cdot\frac{D}{RS}\right)\right)$$

or just

$$A\phi_b = f(B\psi) = \wp$$

Einstein settled on $f(x) = e^{-x}$ for the functional relationship between ϕ and ψ. He fit *A* and *B* for natural data, and ended up with *A*=0.465 and *B*=0.391. It turns out that this relationship works pretty well, except at the tail of the graph (ϕ>0.4). This may be because *A* is no longer a constant, because λ_0 isn't constant due to large excess shear. Einstein refined his argument by stating that for high probability the particle wouldn't be expected to settle after only one step ($\lambda_0 D$). He derived a *new* λ:

$$\lambda = \sum_{m=0}^{\infty}(1-\wp)\wp^{m-1}m\lambda_0 = \frac{\lambda_0}{1-\wp}$$

which yields curve (2) in the handout.

The whole Einstein model was revised in 1950 by Brown, who fitted a new curve to Einstein's data, based on $f(x) = x^{-3}$, rather than $f(x) = e^{-x}$. Brown's curve is:

$$\phi = 40\left(\frac{1}{\psi}\right)^3$$

This curve seems to apply well for low values of ψ (*i.e.* high τ_0 and g_b). Einstein's and Brown's curves are often used together, forming the Einstein-Brown bed load equation:

$$\phi = \begin{cases} \dfrac{1}{0.465} e^{-0.391\psi} & \psi \geq 5.5 \\ 40\left(\dfrac{1}{\psi}\right)^3 & \psi \leq 5.5 \end{cases}$$

A few final comments. The Einstein-Brown equation is somewhat more realistic for large ψ (low τ_0) than DuBoys and other excess shear models, because it shows some transport for $\tau < \tau_c$, whereas excess shear models show none. A major drawback of the Einstein-Brown formulation, however, is that there are no bedform effects.