Albert Einstein claimed that he became a physicist because he found sediment transport (in which he was originally trained) too intractable a problem. How typical, then, that his son decided to one-up dad by taking the problem on.

H.A. Einstein, in 1942, tried a very different tack from the excess shear concept. Einstein described a turbulent fluid where forces vary in both space and time; the *probability* of movement of a particle depends on the *probability* that the fluid forces will exceed resistive forces.

So, let’s consider a flat bed:
And let's assume that if a particle moves, it will move a distance $L$, which we can describe in terms of grain diameters, $L = \lambda_0 D$.

From before,

$$q_b = \frac{\frac{g_b}{(\rho_s - \rho)g}}{A_1 D^2}$$

And, from Einstein,

$$q_b = \frac{L}{A_1 D^2} \phi_s \cdot A_2 D^3 = \frac{A_2}{A_1} \lambda_0 D^2 \phi_s$$

so,

$$\frac{\frac{g_b}{(\rho_s - \rho)gD^2}}{A_1} = \frac{A_1}{A_2} = \lambda_0 \phi_s$$

We would like $\phi_s$ to be independent of time.

$$\phi = t \phi_s$$

Einstein assumes $t \propto \frac{D}{w_s}$, which is the time it takes for a grain to fall one diameter. From long ago,

$$w_s^2 = \frac{4g}{3C_o} \left( \frac{\rho_s - \rho}{\rho} \right)$$
\[ D = \frac{1}{w_s} \sqrt{\frac{4}{3} C_D} \sqrt[4]{\frac{D\rho}{g(\rho_s - \rho)}} \]

and

\[ t = \frac{A_3}{F} \sqrt{\frac{D\rho}{g(\rho_s - \rho)}} \]

where \( F = \sqrt[3]{\frac{4}{3} C_D} \) for spheres, and is given implicitly by Rubey (1933) to be:

\[ F = \sqrt{\frac{2 + \frac{36\mu^2}{gD^3\rho(\rho_s - \rho)}}{3} - \frac{36\mu^2}{gD^3\rho(\rho_s - \rho)}} \]

Now we can formulate \( \varphi \).

\[ \varphi = \frac{A_1A_3}{\lambda_0 A_2} \left[ \frac{1}{F} \frac{g}{(\rho_s - \rho)} \sqrt{\frac{1}{\rho_s - \rho}} \frac{1}{g^2 D^2} \right] \]

Additionally, Einstein assumes that the probability will be some function of particle weight over the average lift on the particle. So,

\[ \varphi = f \left( \frac{A_2D^3(\rho_s - \rho)g}{A_4D^2 \rho V^2} \right) \]

Which \( V \) to choose? \( V \) is taken as the velocity at the top of the laminar boundary layer, and is expressed as \( V = 11.6u^* \). This makes our equation:
\[ \varphi = f\left( \frac{A_2 D^3 (\rho_s - \rho) g}{A_4 D^3 (135 g RS)} \right) \]

Simplifying,
\[ \varphi = f\left( \frac{A_2}{135 A_4}, \frac{\rho_s - \rho}{\rho}, \frac{D}{RS} \right) \]

Now, let \( A = \frac{A_1 A_3}{\lambda_0 A_2}, \quad B = \frac{A_2}{135 A_4} \). This makes the whole Einstein equation:
\[ A \left[ \frac{1}{F} \left( \frac{g_b}{(\rho_s - \rho) g} \sqrt{\frac{\rho}{(\rho_s - \rho)}} \frac{1}{\frac{1}{2} \frac{3}{2} g^2 D^2} \right) \right] = f \left( B \left( \frac{\rho_s - \rho}{\rho}, \frac{D}{RS} \right) \right) \]

or just
\[ A \phi_b = f (B \psi) = \varphi \]

Einstein settled on \( f(x) = e^{-x} \) for the functional relationship between \( \phi \) and \( \psi \). He fit \( A \) and \( B \) for natural data, and ended up with \( A = 0.465 \) and \( B = 0.391 \). It turns out that this relationship works pretty well, except at the tail of the graph (\( \phi > 0.4 \)). This may be because \( A \) is no longer a constant, because \( \lambda_0 \) isn’t constant due to large excess shear. Einstein refined his argument by stating that for high probability the particle wouldn’t be expected to settle after only one step (\( \lambda_0 D \)). He derived a new \( \lambda \):
\[ \lambda = \sum_{m=0}^{\infty} (1 - \varphi) \varphi^{m-1} m \lambda_0 = \frac{\lambda_0}{1 - \varphi} \]

which yields curve (2) in the handout.
The whole Einstein model was revised in 1950 by Brown, who fitted a new curve to Einstein’s data, based on $f(x) = x^{-3}$, rather than $f(x) = e^{-x}$. Brown’s curve is:

$$\phi = 40 \left(\frac{1}{\psi}\right)^3$$

This curve seems to apply well for low values of $\psi$ (i.e. high $\tau_0$ and $g_b$). Einstein’s and Brown’s curves are often used together, forming the Einstein-Brown bed load equation:

$$\phi = \begin{cases} 
\frac{1}{0.465} e^{-0.391\psi} & \psi \geq 5.5 \\
40 \left(\frac{1}{\psi}\right)^3 & \psi \leq 5.5
\end{cases}$$

A few final comments. The Einstein-Brown equation is somewhat more realistic for large $\psi$ (low $\tau_0$) than DuBoys and other excess shear models, because it shows some transport for $\tau < \tau_c$, whereas excess shear models show none. A major drawback of the Einstein-Brown formulation, however, is that there are no bedform effects.