Lecture 12—The Richards Equation and Green-Ampt Infiltration

We've looked at two fairly empirical ways of determining infiltration rate. Unfortunately, each of them requires some sort of parameter with little physical meaning, and therefore a lot of messing around for each field area. What we'd *like* is a method that allows us to use data we either can get easily, or which we already have.

Turns out we've not the first to worry about this. One approach was to make an equation of what the author *thought* was happening, and then tried to solve the equation. Here's the concept...

A guy named Richard thought this one up. Consider Darcy's Law:

$$q = -K\frac{\partial h}{\partial z}$$

Where q is the water discharge, K is the hydraulic conductivity of the soil, and $\frac{\partial h}{\partial z}$ is the hydraulic gradient. You remember this from groundwater, right? While we're here, though, remember our canon—if the water goes in, and doesn't come out, then it's still there. What this means in soils is that any change in the discharge with depth has to be balanced with a change in the soil moisture. Like this:

$$\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} = 0$$

where θ is the moisture content of the soil. This is a simple statement of continuity, just like we had before. If water goes in (through discharge), it increases the soil moisture. Because q is positive downwards, there's a little oddness with the minus sign, but that's it.

Ok! So, we could combine these two equations:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right)$$

There's only one added bit of weirdness. In groundwater, we're used to the flow being driven by hydraulic gradient—basically the slope of the potentiometric surface. *Here*, flow is being driven by two things—the pressure of the overlying water (basically, *z*), PLUS the "suck" of capillary action drawing the water deeper into the soil (let's call this ψ). SO, for our purposes, $h = z + \psi$, and

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(K \frac{\partial (z + \psi)}{\partial z} \right)$$

Or

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(K \frac{\partial z}{\partial z} + K \frac{\partial \psi}{\partial z} \right)$$

Ok, so $\frac{\partial z}{\partial z} = 1$, right? And often people cut $\frac{\partial \psi}{\partial z}$ into two pieces, $\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial \theta} \cdot \frac{\partial \theta}{\partial z}$, so…

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(K + K \frac{\partial \psi}{\partial \theta} \cdot \frac{\partial \theta}{\partial z} \right)$$

Why cut $\frac{\partial \psi}{\partial z}$ into two pieces? It turns out that many people keep track of something called *soil water diffusivity* (*D*), that combines conductivity and capillary pressure:

$$D = K \frac{\partial \psi}{\partial \theta}$$

SOOOOO,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(K + D \frac{\partial \theta}{\partial z} \right)$$

Which looks nice and pretty. We *got* here, remember, by applying Darcy's law in the vertical direction, and recasting discharge as a change in the amount of soil moisture. Because some guy named Richard thought this up, it's called Richard's equation.

Ok. This is the actual important part. Remember how I talked about different *kinds* of equations earlier? This is where is comes home to roost. Horton's equation is one kind of equation—it's the kind you can plug in your numbers for the variables, and you can get an answer. It's the kind of thing we, as geologists, like. HOWEVER, there's a whole separate kind of equations—these *aren't* equations that you just put your values in, and get an answer, they're a kind of shorthand statement of how the world works. They're sort of an equation to *get* an equation. This particular one, if we could solve it, could be used to *make* an equation for infiltration.

The problem is we can't solve it. At least not directly. We *could* if we had a computer simulation, but we can't just make an equation that answers this problem we set up. When scientists run up against this problem, they start making simplifications, and hope that the simplifications enable a solution to the equation. One example here would be to take Richard's equation, and say that *K* and *D* are not functions of *θ* (which they are). IF they aren't, then the equation reduces to:

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial z^2},$$

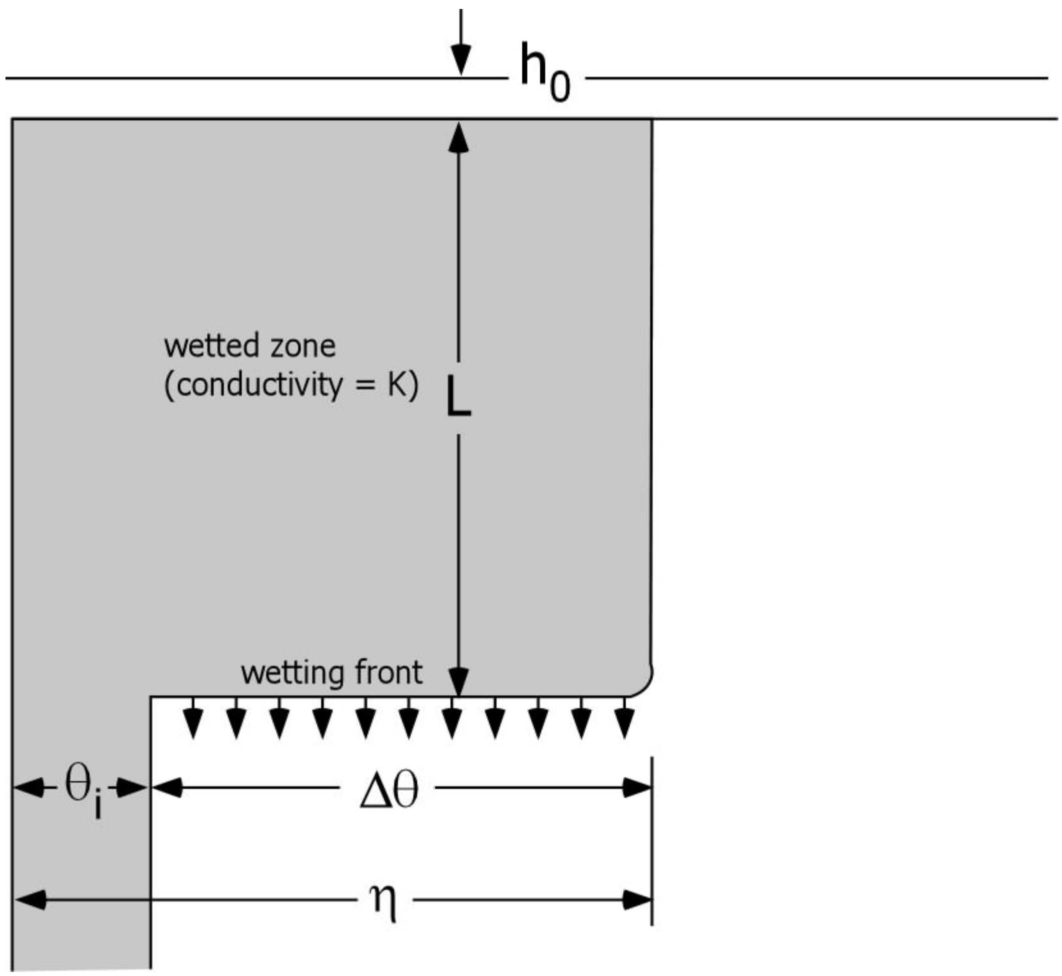
which is solvable. In fact, the answer to this equation *is* Horton's equation. Yeesh, doesn't anyone have an original idea?

Another way to approach this is to simplify the way you set up the problem—basically, you know the world doesn't really work this way, but it makes it easier to get an answer. Such a simplification to Richard's equation was made by Green and Ampt, who decided it's easier to think of infiltration as a solid "front" of wetting moving downward.

Green-Ampt infiltration

Horton's equation and the φ-index method have the same basic drawback—they each require one parameter that's hard to come by, or has to be empirically determined for each watershed. We'd *like* an equation that only has parameters that we can determine easily for soils. Such an equation actually exists! But, it's not as pleasant as Horton. Here's the deal…

Consider a vertical column of soil:



It has some initial amount of moisture in it (θ_i). As water falls on it, a sharp line develops between "dry" soil (with moisture content θ_i) and "wet" soil (with moisture content equal to the porosity of the soil, η). So, if you wanted to know how much water got added to the soil with time (let's call that $F(t)$), the answer would be:

$$F(t) = L(\eta - \theta_i)$$

Where L is the depth of the wetting front.

For ease of use, let's just define a variable that means "the difference between the porosity and the initial moisture content"

$$\Delta\theta \equiv (\eta - \theta_i)$$

So, $F(t) = L\Delta\theta$

Equally, we could take Darcy's Law:

$$q = -K\frac{\partial h}{\partial z}$$

And say that the discharge straight down is effectively the infiltration rate, *f*. Because *q* is defined as positive up, and *f* is defined as positive down, we can make a simple substitution:

$$f = K\frac{\partial h}{\partial z}$$

Here, we're going to step from the differential equation (that we don't like), to a less continuous version:

$$f = K\frac{h_1 - h_2}{z_1 - z_2}$$

Where the subscript "1" means at the surface, and "2" means just on the dry side of the wetted front. The *z* part is easy—at the surface, $z_1=0$, and at the wetted front, $z_2=-L$, so

$$f = K\frac{h_1 - h_2}{0 - -L} = K\frac{h_1 - h_2}{L}$$

We're only left with h_1 and h_2. At the surface, there's no capillary action, and all we're going to worry about is any pressure from ponded water, so $h_1=h_0$, where h_0 is the depth of ponded water. At *depth*, h_2 will be the sum of the pressure from the overlying water (*L*) and capillary suction (ψ), so $h_2=-L-\psi$, and

$$f = K\frac{h_0 - (-L - \psi)}{L} = K\frac{h_0 + L + \psi}{L} \approx K\frac{L + \psi}{L}$$

If h_0 is really small (we can fix it if it isn't, but it's easier if it is).

Ok, next bit. We know that $F(t) = L\Delta\theta$, so let's put in $\frac{F}{\Delta\theta}$ wherever there's an *L*.

$$f = K\left[\frac{\psi\Delta\theta + F}{F}\right]$$

AND, the infiltration *rate* (*f*) is just the derivative of the cumulative amount of infiltration (*F*), so $f = \frac{dF}{dt}$

So,

$$\frac{dF}{dt} = K\left[\frac{\psi\Delta\theta + F}{F}\right]$$

Hey! This is actually relatively easy to integrate! Yay! We could cross multiply to make:

$$Kdt = \left[\frac{\psi\Delta\theta + F}{F}\right]dF$$

And integrate to make:

$$F(t) - \psi\Delta\theta \ln\left(1 + \frac{F(t)}{\psi\Delta\theta}\right) = Kt$$

Which is the infamous Green-Ampt equation. The cool thing is, though, that what we want (*F* or *f*) is a function of only things we can figure out (porosity, initial moisture content, soil conductivity, and soil capillary pressure). The *problem* is that you can't easily put *F* on one side, and all the other stuff on the other. This inability to separate the equation means that the equation is *nonlinear*. There are a few solutions—one is to *make* it linear—if you claim that $\frac{F}{\psi\Delta\theta}$ will be very small, then you could *claim* that $\ln\left(1+\frac{F}{\psi\Delta\theta}\right)\approx\ln(1)=0$

So $F(t)=Kt$. This is not necessarily a good idea, but if you take a nonlinear equation and force it to be linear, then you have *linearized* the equation.

Another option is to set it up with *F* on both sides, like this:

$$F = Kt + \psi\Delta\theta\ln\left(1+\frac{F}{\psi\Delta\theta}\right)$$

Then pick a value for *F*, put it in the right side of the equation, get an answer for *F* that you put *back* into the right side of the equation, which gives you a new *F*…Keep doing this until you don't get any change. This is called *iteration*, and it is a fairly common way of solving nonlinear equations (as an aside—Excel can iterate! I'll show you). So, how do you know where to start from when making a guess for *F*? A good starting point is the *linearized* answer: *Kt*.

There are tables and tables of data on the parameters you want. Here's an example:

TABLE 4.3.1
Green-Ampt infiltration parameters for various soil classes

Soil class	Porosity η	Effective porosity θ_e	Wetting front soil suction head ψ (cm)	Hydraulic conductivity K (cm/h)
Sand	0.437 (0.374–0.500)	0.417 (0.354–0.480)	4.95 (0.97–25.36)	11.78
Loamy sand	0.437 (0.363–0.506)	0.401 (0.329–0.473)	6.13 (1.35–27.94)	2.99
Sandy loam	0.453 (0.351–0.555)	0.412 (0.283–0.541)	11.01 (2.67–45.47)	1.09
Loam	0.463 (0.375–0.551)	0.434 (0.334–0.534)	8.89 (1.33–59.38)	0.34
Silt loam	0.501 (0.420–0.582)	0.486 (0.394–0.578)	16.68 (2.92–95.39)	0.65
Sandy clay loam	0.398 (0.332–0.464)	0.330 (0.235–0.425)	21.85 (4.42–108.0)	0.15
Clay loam	0.464 (0.409–0.519)	0.309 (0.279–0.501)	20.88 (4.79–91.10)	0.10
Silty clay loam	0.471 (0.418–0.524)	0.432 (0.347–0.517)	27.30 (5.67–131.50)	0.10
Sandy clay	0.430 (0.370–0.490)	0.321 (0.207–0.435)	23.90 (4.08–140.2)	0.06
Silty clay	0.479 (0.425–0.533)	0.423 (0.334–0.512)	29.22 (6.13–139.4)	0.05
Clay	0.475 (0.427–0.523)	0.385 (0.269–0.501)	31.63 (6.39–156.5)	0.03

The numbers in parentheses below each parameter are one standard deviation around the parameter value given. *Source:* Rawls, Brakensiek, and Miller, 1983.

It's worth noting that Horton is the equation we often talk about, because it's relatively easy to sketch. It's *not*, however, the equation that tends to get used in hydrology models. Green-Ampt is a nice parametric equation that happens to be unpleasant to work with—Horton is a crappy parametric equation that happens to be pleasant to work with. Computers don't make parameters easier, but they *do* make math easier. As a result, Green-Ampt has experienced a renaissance with the explosion of computing power over the last few decades.