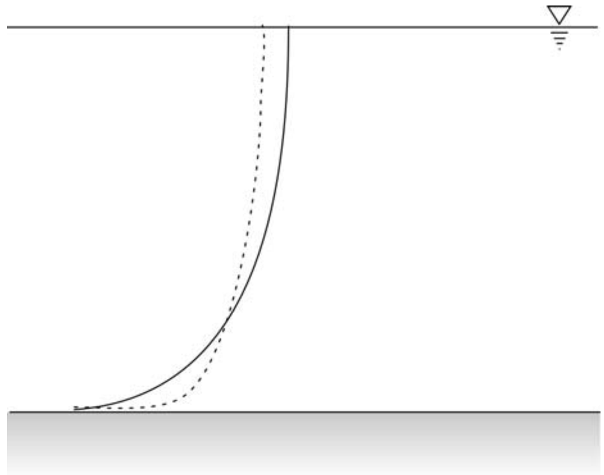
Lecture 16—The law of the wall

How does velocity change with depth?



From last time,

$$\tau = \mu \frac{du}{dy} + Turbulence$$

So how do we define turbulent viscosity? It turns out that horizontal turbulent velocity fluctuations are approximately equal to the distance momentum is transferred across times the velocity gradient:

$$u' \approx \ell \frac{d\bar{u}}{dy}$$

where ℓ is the *mixing length*, defined as the characteristic length across which momentum is transferred in the flow.

What about the shear stress acting on two adjacent layers in the flow? The *force* would be related to the velocity difference between the layers and the rate of mass transfer (which is a function of: how much mass is being transferred (ρ), the *vertical* turbulent velocity component (v'), and the area of transfer (A). The formula looks like this:

$$F = \rho v' A \left(\ell \frac{d\bar{u}}{dy} \right)$$

Remember that a shear stress is just force divided by area, so:

$$\tau_t = \rho v' \left(\ell \frac{d\bar{u}}{dy} \right)$$

Now for the tricky part. Because there's only so much fluid to be had, any fluid going *up* in *v* is related to a loss in horizontal velocity *u*. So:

$$v' = -u'$$

So combining these,

$$\tau_t = \rho \left| \ell \frac{d\bar{u}}{dy} \right| \ell \frac{d\bar{u}}{dy} = \rho \ell^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy}$$

Remember also that we said,

$$\tau = \tau_0 \left(1 - \frac{y}{h} \right)$$

So for turbulent flow,

$$\rho \ell^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} = \tau_0 \left(1 - \frac{y}{h} \right)$$

Minor problem. ℓ varies with distance from the bed. Mercifully, it probably varies linearly, so:

$$\ell = \kappa y$$

and,

$$\rho \kappa^2 y^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} = \tau_0 \left(1 - \frac{y}{h} \right)$$

Where $y/h << 1$ (that's right near the bed),

$$\rho \kappa^2 y^2 \left(\frac{d\bar{u}}{dy} \right)^2 = \tau_0$$

Rearranging,

$$\frac{d\bar{u}}{dy} = \sqrt{\frac{\tau_0}{\rho}}\frac{1}{\kappa y}$$

And integrating,

$$u = \sqrt{\frac{\tau_0}{\rho}}\frac{1}{\kappa}\ln y + \text{constant}$$

Finally, we define $u_* \equiv \sqrt{\frac{\tau_0}{\rho}}$, so

$$u = u_* \frac{1}{\kappa}\ln y + \text{constant}$$

Unfortunately, we can't enforce $u(0)=0$ because ln(0) doesn't exist. This is a problem for a velocity distribution that's supposed to talk about what happens close to the bed.

HOWEVER, we could *define* some length scale, y_0, at which u approaches 0, and set $u(y_0)=0$.

THEN, our equation becomes,

$$u = u_* \frac{1}{\kappa}\ln\left(\frac{y}{y_0}\right)$$

which is the turbulent velocity profile near the bed. It is better known as *the law of the wall*.

So what's κ? It's called *von Karman's constant* and has been empirically determined to be 0.41

What about y_0? We could figure it out by measuring velocities empirically. It's typically very small—smaller than the roughness elements on the bed. It's commonly taken as a function of roughness elements on the bed, in fact:

$y_0 = \frac{D_{84}}{30}$, where D_{84} is the size below which 84% of the particles on the bed reside (1σ above the median).

As a cautionary note, this only works on a planar bed (so that sand grains are the only roughness elements, and not bedforms), and with well-sorted sediments. Note that it also ignores other forms of roughness, like plants and meandering.

With these caveats, our velocity equation becomes:

$$\frac{u}{u_*} = 5.75\log\left(\frac{y}{D_{84}}\right) + 8.5$$

Often, the law of the wall is applied as if it were valid throughout the flow (remember, it's not). We can get away with this, though, because higher up in the flow, velocity doesn't change much with depth.

Last up, we need to define an *average* velocity for the law of the wall. It's

$$U = \frac{1}{(h - y_0)}\int_{y_0}^{h} \overline{u}(y)dy = \frac{u_*}{\kappa}\left[\ln\left(\frac{h}{y_0}\right) - \left(1 - \frac{y_0}{h}\right)\right]$$

which is a function only of relative roughness.

Last worry. Remember how at the beginning of all this we had the expression,

$$\tau = \mu\frac{du}{dy} + Turbulence\ ?$$

This entire analysis has been based on a new definition of turbulent shear stress. But we have *two* components of shear stress—one turbulent and one viscous. To be correct, we need to include *both* components in our velocity distribution:

$$\tau = \tau_v + \tau_t = \mu\frac{d\overline{u}}{dy} + \rho\ell^2\left(\frac{d\overline{u}}{dy}\right)^2$$

Dealing with *this* little annoyance will be the subject of our next lecture.