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THE VALUE OF BANACH LIMITS ON A CERTAIN SEQUENCE OF ALL RATIONAL NUMBERS IN THE INTERVAL $(0, 1)$

(*Dedicated to the Memory of Professor Joe Diestel*)

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Abstract

In this article, we show that a certain sequence r of all rational numbers in the interval $(0, 1)$:

$$r := \{r(j)\}$$

$$= \left\{ \underbrace{\frac{1}{2}}_{\text{1st block}}, \underbrace{\frac{1}{3}, \frac{2}{3}}_{\text{2nd block}}, \underbrace{\frac{1}{4}, \frac{3}{4}}_{\text{3rd block}}, \ldots, \underbrace{\frac{1}{m+1}, \ldots, \frac{k}{m+1}, \ldots, \frac{m}{m+1}}_{m\text{th block}}, \ldots \right\},$$

where $(k, m+1) = 1$, is an almost convergent sequence in l^{∞}, and its value of Banach limits $L(r) = 1/2$ for all $L \in (l^{\infty})^{*}$.

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Let l^{∞} be the Banach space of bounded sequences $x := \{x(n)\}_{n=1}^{\infty}$ of real numbers with norm $\| x \|_{\infty} = \sup |x(n)|$. A *Banach limit L* is a linear and bounded functional on l^{∞}, which satisfies the three conditions:

(a) if $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and $x(n) \geq 0$, $n = 1, 2, \ldots,$ then $L(x) \geq 0$;

(b) if $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$, and T is the *translation operator*: $Tx := \{x(2), x(3), \ldots\}$, then $L(x) = L(Tx)$;

(c) $L(1) = 1$, where $1 := \{1, 1, \ldots\}$.

We know [4, p. 310] that there are infinitely many Banach limits in $(l^{\infty})^*$, the dual space of l^{∞}. Thus, it does not make sense to speak of finding a particular value for Banach limits of a sequence $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$, because normally the different Banach limits are different functionals. It is, however, interesting that there are some elements in l^{∞} for which the values of all Banach limits are the same. For example, $L(x) = \lim_{n \to \infty} x(n)$ for any Banach limit L, if x is an element of c, where c is the Banach space of convergent sequences of real numbers with the superior norm. Furthermore, this phenomenon can happen on some elements of $l^{\infty} \backslash c$. Let

$$a := \{1, \underbrace{0, \ldots, 0}_{(m-1)\text{-times}}, 1, \underbrace{0, \ldots, 0}_{(m-1)\text{-times}}, 1, \ldots\}.$$

Property (b) of Banach limits implies that for any Banach limit L,

$$L(a) = L(Ta) = \cdots = L(T^{m-1}a);$$

so by linearity and property (c) of Banach limits, we have that

$$L(a) + L(Ta) + \cdots + L(T^{m-1}a) = L(1) = 1.$$

Hence

$$L(a) = L(Ta) = \cdots = L(T^{m-1}a) = \frac{1}{m}.$$

Moreover, if

$$b := \{\underbrace{1, ..., 1}_{k\text{-}times}, \underbrace{0, ..., 0}_{(m-k)\text{-}times}, \underbrace{1, ..., 1}_{k\text{-}times}, \underbrace{0, ..., 0}_{(m-k)\text{-}times}, ...\},$$

then

$$L(b) = L(a) + L(T^{m-1}a) + \cdots + L(T^{m-k+1}a) = \frac{k}{m}.$$

In [2], Lorentz called a sequence $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ *almost convergent*, if its all Banach limit values $L(x)$ are the same for $L \in (l^{\infty})^{*}$. In this case, we call $L(x)$ the *F-limit* of *x*. In his paper, Lorentz proved the following main result:

Theorem 1 (Lorentz [2, Theorem 1])**.** *A sequence* $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ *is almost convergent with F-limit* $L(x)$ *if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=i}^{i+n-1} x(t) = L(x)$$

uniformly in i.

This Lorentz theorem offers a way to find values of Banach limits for almost convergent sequences in l^{∞}. Based on Lorentz [2] and Sucheston [4], we give another way [1] to find the value of Banach limits of *x*, when *x* is an almost convergent sequences in l^{∞}.

Recalling some concepts, we created in [1].

Definition 1. A real number *a* is said to be a *sub-limit* of the sequence $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$, if there exists a subsequence $\{x(n_k)\}_{k=1}^{\infty}$ of *x* with limit *a*. The set of all sub-limits of *x* is denoted by $S(x)$ and the set of all limit points of $S(x)$ is denoted by $S'(x)$.

Definition 2. Suppose $a \in S(x)$ for element $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$. A subsequence $\{x(n_k)\}_{k=1}^{\infty}$ of *x* is called an *essential subsequence* of *a* if it converges to *a*, and for any subsequence $\{x(m_t)\}_{t=1}^{\infty}$ of *x* with limit *a*, except finite entries, all its entries are entries of $\{x(n_k)\}_{k=1}^{\infty}$.

Definition 3. Let $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and let $\{x(n_k)\}_{k=1}^{\infty}$ be a subsequence of *x*. Define

$$w^{u}(\{x(n_k)\}) = \limsup_{n\to\infty}\left(\sup_{i} \frac{A(\{k : i \leq n_k \leq i + n - 1\})}{n}\right)$$

and

$$w_{l}(\{x(n_k)\}) = \liminf_{n\to\infty}\left(\inf_{i} \frac{A(\{k : i \leq n_k \leq i + n - 1\})}{n}\right),$$

where $A(E)$ is the number of elements of set *E*. $w^{u}(\{x(n_k)\})$ and $w_{l}(\{x(n_k)\})$ are called the *upper weight* and *lower weight* of the subsequence $\{x(n_k)\}_{k=1}^{\infty}$, respectively. If $w^{u}(\{x(n_k)\}) = w_{l}(\{x(n_k)\})$, then the subsequence $\{x(n_k)\}_{k=1}^{\infty}$ is said to be *weightable* and the weight of $\{x(n_k)\}_{k=1}^{\infty}$ is denoted by $w(\{x(n_k)\})$, and

$$w(\{x(n_k)\}) = w^{u}(\{x(n_k)\}) = w_{l}(\{x(n_k)\}).$$

We verified [1, Theorem 1] that all essential subsequences of *a*, $a \in S(x)$, have the same upper weight and lower weight, respectively. They are called the *upper weight* and *lower weight* of *a*, and denoted by $w^{u}(a)$ and $w_{l}(a)$, respectively. The *weight* of *a* is denoted by $w(a)$, if $w^{u}(a) = w_{l}(a)$. We have the following main results.

Theorem 2 [1, Theorem 4]**.** *Suppose* $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ *and the set of all sub-limits of x,* $S(x) = \{a_1, a_2, ..., a_m\}$ *is finite, where* $a_i \neq a_j$, *if* $i \neq j$. *If* $w(a_t)$ *exists for each t,* $1 \leq t \leq m$, *then x is almost convergent and for any Banach limit* $L \in (l^{\infty})^*$,

$$L(x) = \sum_{t=1}^{m} a_t w(a_t).$$

We see that for almost convergent sequence, the value of Banach limits only dependent on the sub-limit points and their weights. From Theorem 2, we obtain a familiar formula.

Corollary 1 [1, Corollary 2]**.** *For a given positive integer m, let*

$$x = \{x_1(1), ..., x_m(1), x_1(2), ..., x_m(2), ..., x_1(n), ..., x_m(n), ...\},$$

where for each t, $\lim_{n\to\infty} x_t(n) = a_t$, $1 \leq t \leq m$. *Then x is almost convergent and for any Banach limit* $L \in (l^{\infty})^*$,

$$L(x) = \frac{a_1 + a_2 + \cdots + a_m}{m}.$$

We actually proved a more general result as below.

Theorem 3 [1, Theorem 6]**.** *Suppose* $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ *and* $S(x)$ *is infinite but countable and* $S'(x)$, *the set of limit points of* $S(x)$, *is a nonempty finite set. If* $w(a)$ *exists for all* $a \in S(x)$, *then x is almost convergent and for any Banach limit* $L \in (l^{\infty})^*$,

$$L(x) = \sum_{a \in S(x)} aw(a).$$

Natural question can be asked that does there exist an almost convergent sequence $x \in l^{\infty}$ for which $S'(x)$ is an infinite set? The answer is ‘Yes’. We proved more that $S'(x)$ can be uncountable infinite in next Theorem 4 of this article.

We need some preliminary knowledge. Suppose m is a positive integer. We define $\varphi(m)$ to be the number of integers k, $1 \le k < m$ such that $(k, m) = 1$, which denotes that k and m are relative primes. The function φ is called the *Euler phi function* [3, p. 54]. It is well known in number theory that if $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where the p's are distinct primes and α_i, $1 \le i \le t$, are positive integers, then [3, p. 58, Theorem 2.7]

$$\varphi(m) = \prod_{j=1}^{t} (p_j - 1) p_j^{\alpha_j - 1}.$$

We introduce the following lemmas that we were unable to find in the literatures.

Lemma 1. *For any positive integer m,*

$$\frac{1}{m} \sum_{(k, m)=1, 1 \le k < m} k = \frac{1}{2} \varphi(m).$$

Proof. Suppose that $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. Note that there are $p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} - 1$ numbers containing the factor of p_1 among the set of $\{1, 2, 3, ..., m - 1\}$:

$$p_1, 2p_1, 3p_1, ..., (p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} - 1) p_1,$$

there are $p_1^{\alpha_1} p_2^{\alpha_2 - 1} \cdots p_t^{\alpha_t} - 1$ numbers containing the factor of p_2 among the set of $\{1, 2, 3, ..., m - 1\}$:

$$p_2, 2p_2, 3p_2, ..., (p_1^{\alpha_1} p_2^{\alpha_2 - 1} \cdots p_t^{\alpha_t} - 1) p_2,$$

…, and there are $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t - 1} - 1$ numbers containing the factor of p_t among the set of $\{1, 2, 3, ..., m - 1\}$:

$$p_t,\ 2p_t,\ 3p_t,\ ...,\ (p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t-1}-1)\,p_t.$$

The sums of these t groups are

$$\frac{1}{2}\,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}(p_1^{\alpha_1-1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}-1)=\frac{1}{2}\,m(p_1^{\alpha_1-1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}-1),$$

$$\frac{1}{2}\,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}(p_1^{\alpha_1}p_2^{\alpha_2-1}\cdots p_t^{\alpha_t}-1)=\frac{1}{2}\,m(p_1^{\alpha_1}p_2^{\alpha_2-1}\cdots p_t^{\alpha_t}-1),$$

$$\vdots$$

$$\frac{1}{2}\,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t-1}-1)=\frac{1}{2}\,m(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t-1}-1),$$

respectively.

Similarly, there are $p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_t^{\alpha_t}-1$ numbers containing the factor of p_1p_2 among the set of $\{1,\ 2,\ 3,\ ...,\ m-1\}$:

$$p_1p_2,\ 2p_1p_2,\ 3p_1p_2,\ ...,\ (p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_t^{\alpha_t}-1)\,p_1p_2,$$

…, and there are $p_1^{\alpha_1}\cdots p_{t-1}^{\alpha_{t-1}-1}p_t^{\alpha_t-1}-1$ numbers containing the factor of $p_{t-1}p_t$ among the set of $\{1,\ 2,\ 3,\ ...,\ m-1\}$:

$$p_{t-1}p_t,\ 2p_{t-1}p_t,\ 3p_{t-1}p_t,\ ...,\ (p^{\alpha_1}\cdots p_{t-1}^{\alpha_{t-1}}p_t^{\alpha_t}-1)\,p_{t-1}p_t.$$

The sums of these $\binom{t}{2}$ groups are

$$\frac{1}{2}\,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}(p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_t^{\alpha_t}-1)=\frac{1}{2}\,m(p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_t^{\alpha_t}-1),$$

$$\vdots$$

$$\frac{1}{2}\,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t-1}-1)=\frac{1}{2}\,m(p_1^{\alpha_1}\cdots p_{t-1}^{\alpha_{t-1}-1}p_t^{\alpha_t-1}-1),$$

respectively, and so on.

Finally, there are $p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_t^{\alpha_t-1}-1$ numbers containing the factor of $p_1p_2\cdots p_t$ among the set of $\{1, 2, 3, ..., m-1\}$:

$$p_1p_2\cdots p_t,\ 2p_1p_2\cdots p_t,\ ...,\ (p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_t^{\alpha_t-1}-1)\,p_1p_2\cdots p_t.$$

The sum of this group is

$$\frac{1}{2}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}(p_1^{\alpha_1-1}\cdots p_t^{\alpha_t-1}-1)=\frac{1}{2}m(p_1^{\alpha_1-1}\cdots p_t^{\alpha_t-1}-1).$$

Thus,

$$\sum_{(k,m)=1,1\le k<m} k$$

$$=\frac{m}{2}(m-1)-\frac{m}{2}\sum_{j=1}^{t}\left(p_j^{\alpha_j-1}\prod_{i\ne j}p_i^{\alpha_i}-1\right)$$

$$+\frac{m}{2}\sum_{1\le i<j\le t}\left(p_i^{\alpha_i-1}p_j^{\alpha_j-1}\prod_{k\ne i,\,j}p_k^{\alpha_k}-1\right)$$

$$+\cdots+(-1)^{t-1}\frac{m}{2}\sum_{j=1}^{t}\left(p_j^{\alpha_j}\prod_{i\ne j}p_i^{\alpha_i-1}-1\right)+(-1)^{t}\frac{m}{2}\left(\prod_{j=1}^{t}p_j^{\alpha_j-1}-1\right)$$

$$=\frac{m}{2}\left[\left(\prod_{j=1}^{t}p_j^{\alpha_j}-1\right)-\sum_{j=1}^{t}\left(p_j^{\alpha_j-1}\prod_{i\ne j}p_i^{\alpha_i}-1\right)\right.$$

$$+\sum_{1\le i<j\le t}\left(p_i^{\alpha_i-1}p_j^{\alpha_j-1}\prod_{k\ne i,\,j}p_k^{\alpha_k}-1\right)$$

$$\left.+\cdots+(-1)^{t-1}\sum_{j=1}^{t}\left(p_j^{\alpha_j}\prod_{i\ne j}p_i^{\alpha_i-1}-1\right)+(-1)^{t}\left(\prod_{j=1}^{t}p_j^{\alpha_j-1}-1\right)\right]$$

$$= \frac{m}{2}\Bigg[\prod_{j=1}^{t} p_j^{\alpha_j} - \sum_{j=1}^{t} p_j^{\alpha_j - 1}\prod_{i \neq j} p_i^{\alpha_i}$$

$$+ \sum_{1 \leq i < j \leq t} p_i^{\alpha_i - 1} p_j^{\alpha_j - 1} \prod_{k \neq i,\, j} p_k^{\alpha_k} + \cdots$$

$$+ (-1)^{t-1}\sum_{j=1}^{t} p_j^{\alpha_j}\prod_{i \neq j} p_i^{\alpha_i - 1} + (-1)^t \prod_{j=1}^{t} p_j^{\alpha_j - 1} - (1-1)^t\Bigg]$$

$$= \frac{m}{2}\prod_{j=1}^{t} p_j^{\alpha_j - 1}\Bigg[\prod_{j=1}^{t} p_j - \sum_{j=1}^{t}\prod_{i \neq j} p_i$$

$$+ \sum_{1 \leq i < j \leq t}\ \prod_{k \neq i,\, j} p_k + \cdots + (-1)^{t-1}\sum_{j=1}^{t} p_j + (-1)^t\Bigg]$$

$$= \frac{m}{2}\prod_{j=1}^{t} p_j^{\alpha_j - 1}\prod_{j=1}^{t}(p_j - 1) = \frac{m}{2}\varphi(m).$$

Hence,

$$\frac{1}{m}\sum_{(k,\, m)=1,\, 1 \leq k < m} k = \frac{1}{2}\varphi(m). \qquad \square$$

Lemma 2. *Suppose that* m *and* l *are positive integers with* $l > 1$. *Let* $n = \sum_{j=m+1}^{m+l} \varphi(j)$. *Then*

$$\lim_{l \to \infty} \frac{l}{n} = 0.$$

Proof. We know that [3, p. 228, Theorem 6.21]

$$\sum_{j=1}^{m} \varphi(j) = \frac{3m^2}{\pi^2} + O(m \ln m),$$

where $f = O(g)$ means that there exists a constant $c > 0$ such that $|f(x)| \leq cg(x)$ for all x in the intersection of domains of f and g. Thus,

$$n = \sum_{j=m+1}^{m+l} \varphi(j) = \frac{3(m+l)^2 - 3m^2}{\pi^2} + O((m+l)\ln(m+l)) - O(m \ln m)$$

$$= \frac{6ml + 3l^2}{\pi^2} + O((m+l)\ln(m+l)) - O(m \ln m),$$

which implies $n = O(l^2)$. Hence

$$\lim_{l \to \infty} \frac{l}{n} = 0. \qquad \square$$

Now we back to our goal of the following result.

Theorem 4. *Define an element of* l^∞ *as follows*:

$r := \{r(j)\}$

$$= \left\{ \underbrace{\frac{1}{2}}_{1st\ block}, \underbrace{\frac{1}{3}, \frac{2}{3}}_{2nd\ block}, \underbrace{\frac{1}{4}, \frac{3}{4}}_{3rd\ block}, \ldots, \underbrace{\frac{1}{m+1}, \ldots, \frac{k}{m+1}, \ldots, \frac{m}{m+1}}_{mth\ block}, \ldots \right\},$$

where $(k, m+1) = 1$. *Then* $S'(r) = S(r) = [0, 1]$ *is an uncountable set*, *and* r *is an almost convergent sequence in* l^∞ *and also* $L(r) = 1/2$ *for any Banach limit* $L \in (l^\infty)^*$.

Proof. It is well known that $S'(r) = S(r) = [0, 1]$. We just verify the remaining parts of the conclusion. For all positive integers i, there exists an integer m such that

$$\sum_{j=2}^{m} \varphi(j) \leq i < \sum_{j=2}^{m+1} \varphi(j).$$

Let $i = \sum_{j=2}^{m} \varphi(j) + q$, where $0 \le q < \varphi(m+1)$. For any positive integers n and m, m is determined by i above, there is a positive integer l such that

$$\sum_{j=2}^{m+l} \varphi(j) \le i + n - 1 < \sum_{j=2}^{m+l+1} \varphi(j).$$

Let $i + n - 1 = \sum_{j=2}^{m+l} \varphi(j) + s$, where $0 \le s < \varphi(m + l + 1)$. Then

$$\begin{aligned} n &= \sum_{j=2}^{m+l} \varphi(j) + s - i + 1 \\ &= \sum_{j=2}^{m+l} \varphi(j) + s - \left(\sum_{j=2}^{m} \varphi(j) + q \right) + 1 \\ &= \sum_{j=m+1}^{m+l} \varphi(j) + s - q + 1, \end{aligned}$$

from which we have

$$\sum_{j=m+2}^{m+l} \varphi(j) = n - (\varphi(m+1) - q) - s - 1.$$

With the help of Lemma 1 and the fact of $\varphi(m) < m$ for any m, we have

$$\begin{aligned} &\frac{1}{n} \sum_{t=i}^{i+n-1} r(t) \\ &\ge \frac{1}{n} \sum_{t=\sum_{j=2}^{m+1} \varphi(j)+1}^{\sum_{j=2}^{m+l} \varphi(j)} r(t) \end{aligned}$$

$$= \frac{1}{n}\left[\frac{1}{m+2}\sum_{(k,\, m+2)=1,\, 1\le k<m+2} k + \cdots + \frac{1}{k+l}\sum_{(k,\, m+l)=1,\, 1\le k<m+l} k\right]$$

$$= \frac{1}{n}\left[\frac{1}{2}\varphi(m+2) + \cdots + \frac{1}{2}\varphi(m+l)\right] = \frac{1}{2n}\sum_{t=m+2}^{m+l}\varphi(t)$$

$$= \frac{n - (\varphi(m+1) - q) - s - 1}{2n}$$

$$\geq \frac{1}{2} - \frac{(\varphi(m+1) - q) + \varphi(m+l+1) + 1}{2n}$$

$$\geq \frac{1}{2} - \frac{(m+1) + (m+l+1) + 1}{2n}$$

$$= \frac{1}{2} - \frac{2m+l+3}{2n} = \frac{1}{2} - \frac{2m+3}{2n} - \frac{l}{2n},$$

which holds for any positive integers n and any fixed positive integer i. We know that m is fixed when i is fixed, so $\lim_{n\to\infty}\frac{2m+3}{2n} = 0.$ Note that l goes to infinity as n goes to infinity. By Lemma 2, we have

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{t=i}^{i+n-1} r(t)\right) \geq \liminf_{n\to\infty}\left(\frac{1}{2} - \frac{2m+3}{2n} - \frac{l}{2n}\right) = \frac{1}{2}$$

for all $i \in N$.

On the other hand, we have

$$\frac{1}{n}\sum_{t=i}^{i+n-1} r(t)$$

$$\leq \frac{1}{n}\sum_{t=\sum_{j=2}^{m}\varphi(j)+1}^{\sum_{j=2}^{m+l+1}\varphi(j)} r(t)$$

$$= \frac{1}{n}\left[\frac{1}{m+1}\sum_{(k,\, m+1)=1,\, 1\le k<m+1} k + \cdots + \frac{1}{k+l+1}\sum_{(k,\, m+l+1)=1,\, 1\le k<m+l+1} k\right]$$

$$= \frac{1}{n}\left[\frac{1}{2}\varphi(m+1) + \cdots + \frac{1}{2}\varphi(m+l+1)\right]$$

$$= \frac{1}{2n}\sum_{t=m+1}^{m+l+1}\varphi(t) = \frac{n + \varphi(m+l+1) + q - s - 1}{2n}$$

$$= \frac{1}{2} + \frac{(\varphi(m+l+1) - s) + q - 1}{2n}$$

$$\le \frac{1}{2} + \frac{m+l+1+m+1-1}{2n} = \frac{1}{2} + \frac{2m+1}{2n} + \frac{l}{2n},$$

which holds for any positive integer n and any fixed positive integer i. We know that $\lim_{n\to\infty} \frac{2m+1}{2n} = 0$. Using Lemma 2 again, we have

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{t=i}^{i+n-1} r(t)\right) \le \limsup_{n\to\infty}\left(\frac{1}{2} + \frac{2m+1}{2n} + \frac{l}{2n}\right) = \frac{1}{2}$$

for all $i \in N$.

Summarizing the discussion above, we obtain that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=i}^{i+n-1} r(t) = \frac{1}{2}$$

for all $i \in N$. Thus,

$$\varliminf_{n\to\infty}\left(\frac{1}{n}\sum_{t=i}^{i+n-1} r(t)\right) = \varlimsup_{n\to\infty}\left(\frac{1}{n}\sum_{t=i}^{i+n-1} r(t)\right) = \frac{1}{2}.$$

Using Lorentz's formula (6) [2, p.169], we have:

$$\varliminf_{n\to\infty}\left(\frac{1}{n}\sum_{t=i}^{i+n-1} r(t)\right) \le L(r) \le \varlimsup_{n\to\infty}\left(\frac{1}{n}\sum_{t=i}^{i+n-1} r(t)\right)$$

for all Banach limits $L \in (l^{\infty})^{*}$. Hence, $L(r) = \frac{1}{2}$, for all $L \in (l^{\infty})^{*}$ and r is an almost convergent sequence in l^{∞}. □

From the argument of Theorem 4, we have a new corollary of Lorentz' theorem as follows:

Corollary 2. *Suppose* $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$. *If the following limits exist with the same value l,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=i}^{i+n-1} x(t) = l$$

for all positive integers $i \in N$, *then x is an almost convergent sequence in* l^{∞} *and* $L(x) = l$ *for all Banach limits* $L \in (l^{\infty})^{*}$.

Notice. Corollary 2 is a complement of Theorem 6 in [1], which helps us in verifying and finding the value of Banach limits for an almost convergent sequence $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ with $S'(x)$ as an uncountable set.

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