ON DISTRIBUTION AND ALMOST CONVERGENCE OF BOUNDED SEQUENCES

CHAO YOU AND BAO QI FENG

Abstract. In this paper, we give the concepts of properly distributed and simply distributed sequences, and prove that they are almost convergent. Basing on these, we review the work of Feng and Li [Feng, B. Q. and Li, J. L., Some estimations of Banach limits, J. Math. Anal. Appl. 323(2006) No. 1, 481-496. MR2262220 46B45 (46A45).], which is shown to be a special case of our generalized theory.

1. PRELIMINARY AND BACKGROUND

Let l^∞ be the Banach space of bounded sequences of real numbers $x := \{x(n)\}_{n=1}^\infty$ with norm $\|x\|_\infty = \sup |x(n)|$. As an application of Hahn-Banach theorem, a *Banach limit* L is a bounded linear functional on l^∞, which satisfies the following properties:

(a)If $x := \{x(n)\}_{n=1}^\infty \in l^\infty$ and $x(n) \geq 0$, then $L(x) \geq 0$;

(b)If $x := \{x(n)\}_{n=1}^\infty \in l^\infty$ and $Tx = \{x(2), x(3), \ldots\}$, then $L(x) = L(Tx)$, where T is the *translation operator*;

(c)$L(1) = 1$, where $1 := \{1, 1, \ldots\}$;

(d)$\|L\| = 1$;

(e)If $x := \{x(n)\}_{n=1}^\infty \in c$, then $L(x) = \lim_{n\to\infty} x(n)$, where c is the Banach subspace of l^∞ consisting of convergent sequences.

Since the Hahn-Banach norm-preserving extension is not unique, there must be many Banach limits in the dual space of l^∞, and usually different Banach limits have different values at the same element in l^∞. However, there indeed exist sequences whose values of all Banach limits are the same. Condition (e) is a trivial example. Besides that, there also exist nonconvergent sequences satisfying this property, for such examples please see [1] and [2]. In [3], G. G. Lorentz called a sequence $x := \{x(n)\}_{n=1}^\infty$ *almost convergent*, if all Banach limits of x, $L(x)$, are the same. In his paper, Lorentz proved the following criterion for almost convergent sequences:

Theorem 1.1. *A sequence $x := \{x(n)\}_{n=1}^\infty \in l^\infty$ is almost convergent if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=i}^{i+n-1} x(t) = L(x)$$

uniformly in i.

There is no doubt that Lorentz' theorem is a landmark in Banach limit theory, which in theory points out all the almost convergent sequences. Recently, basing

2000 *Mathematics Subject Classification.* Primary 40G05, 46A35, 54A20; Secondary 11K36.

Key words and phrases. Banach limit, almost convergent sequence, properly distributed sequence, simply distributed sequence.

on Lorentz [3] and Sucheston [4], Feng B. Q. and Li J. L. gave another way [1] to find the value of Banach limits of x, where x is an element of the space of almost convergent sequences with some properties. In this paper, we will make a remark on the concept of essential subsequence (Definition 2, [1]), then cite Theorem 4([1]) to develop our theory, and at last use our theory to review two main results in [1], in the bid to include [1] into our framework and show that we have genuinely done a work of generalization in theory. Thus, we'd better make a short introduction to the main results of [1] first, making the notations and terminologies available.

Definition 1.2 (Definition 1, [1])**.** A real number a is said to be a sub-limit of the sequence $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$, if there exists a subsequence $\{x(n_k)\}_{k=1}^{\infty}$ of x with limit a. The set of all sub-limits of x is denoted by $S(x)$ and the set of all limit points of $S(x)$ is denoted by $S'(x)$.

Definition 1.3 (Definition 3, [1])**.** Let $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$, and let $\{x(n_k)\}_{k=1}^{\infty}$ be a subsequence of x. Define

$$w^u(\{x(n_k)\}) = \limsup_{n\to\infty} \left(\sup_i \frac{A(\{k : i \le n_k \le i+n-1\})}{n} \right)$$

and

$$w_l(\{x(n_k)\}) = \liminf_{n\to\infty} \left(\inf_i \frac{A(\{k : i \le n_k \le i+n-1\})}{n} \right),$$

where $A(E)$ is the cardinality of the set E. $w^u(\{x(n_k)\})$ and $w_l(\{x(n_k)\})$ are called the upper and lower weights of the subsequence $\{x(n_k)\}_{k=1}^{\infty}$ respectively. If $w^u(\{x(n_k)\}) = w_l(\{x(n_k)\})$, then the subsequence $\{x(n_k)\}_{k=1}^{\infty}$ is said to be weightable and the weight of $\{x(n_k)\}_{k=1}^{\infty}$ is denoted by $w(\{x(n_k)\})$, and $w(\{x(n_k)\}) = w^u(\{x(n_k)\}) = w_l(\{x(n_k)\})$.

Remark 1.4. It should be emphasized that our Definition 1.3 is slightly different from Definition 3([1]), with $\lim_{n\to\infty}$ there replaced by $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ for $w^u(\cdot)$ and $w_l(\cdot)$ respectively. Such expression is more accurate, since there is no reason to guarantee the existence of $\lim_{n\to\infty}$.

Definition 1.5 (Definition 2, [1])**.** Suppose $a \in S(x)$ for some $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$. A subsequence $\{x(n_k)\}_{k=1}^{\infty}$ of x is called an essential subsequence of a if it converges to a, and for any subsequence $\{x(m_t)\}_{t=1}^{\infty}$ of x with limit a, except finite entries, all its entries are entries of $\{x(n_k)\}_{k=1}^{\infty}$.

Theorem 1.6 (Theorem 1, [1])**.** *Let $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$. Suppose $a \in S(x)$. Let $\{x(n_k)\}_{k=1}^{\infty}$ and $\{x(m_t)\}_{t=1}^{\infty}$ be two essential subsequences of a. Then $w^u(\{x(n_k)\}) = w^u(\{x(m_t)\})$ and $w_l(\{x(n_k)\}) = w_l(\{x(m_t)\})$.*

Theorem 1.6 points out that, for $a \in S(x)$, all essential subsequences of a have the same upper weight and lower weight, respectively. They are called the *upper* and *lower weights* of a in the sequence x, and denoted by $w^u(a)$ and $w_l(a)$, respectively. The *weight* of a in the sequence x is denoted by $w(a)$, if $w^u(a) = w_l(a)$.

We remark that not every sub-limit $a \in S(x)$ has an essential subsequence. The following proposition shows that this happens only when a is an isolated sub-limit of x. This is an important correction to [1], and consideration on this problem directly leads to our present work.

Proposition 1.7. [1] Let $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and suppose $a \in S(x)$. a has an essential subsequence if and only if a is an isolated sub-limit of x.

Proof. If a is an isolated sub-limit of x, then there exists $\varepsilon_0 > 0$ such that $(a-\varepsilon_0, a+\varepsilon_0) \bigcap S(x) = \{a\}$. Let $\{x(n_k)\}$ denote all the terms of x that lying in $(a-\varepsilon_0, a+\varepsilon_0)$, we will show that $\{x(n_k)\}$ is the desired essential subsequence of x. Since $\{x(n_k)\}$ is infinite and bounded, it must have at least one convergent subsequence or sub-limit. But a is an isolated sub-limit, hence $\{x(n_k)\}$ has just one sub-limit, i.e., a. That's to say $\{x(n_k)\}$ is convergent to a. For any subsequence $\{x(m_t)\}$ of x that converging to a, from the definition of $\{x(n_k)\}$ and convergence of $\{x(m_t)\}$ to a, all of the terms of this subsequence under consideration, except finite number of them, must be in $\{x(n_k)\}$. So $\{x(n_k)\}$ is an essential subsequence of a.

Conversely, suppose that a has an essential subsequence $\{x(n_k)\}$. Assume a is not an isolated sub-limit of sequence x, then there exist a sequence of sub-limits $\{a_n\}$ that converges to a. We know, for each a_n from $\{a_n\}$, there is a subsequence $\{x_n^i\}$ that converges to a_n when $i \to \infty$. Without loss of generality, we can assume $0 < d_n = |a - a_n| < 1/n$. Then, for each n, we can find y_n from $\{x_n^i\}$ such that y_n doesn't lie in $\{x(n_k)\}$ and $|y_n - a_n| < 1/n$. Actually, this construction is possible. Since a and a_n are distinct with distance d_n, then we can find positive integer N_1 and N_2 such that, when $k > N_1$, $i > N_2$, it holds that $|x(n_k) - a| < d_n/3$ and $|x_n^i - a_n| < d_n/3$, respectively. It is easy to see such y_n can be found and satisfying $|y_n - a| < d_n < 1/n$. Here we have constructed a subsequence $\{y_n\}$ converging to a, but not lying in the essential subsequence $\{x(n_k)\}$, which leads to a contradiction. □

Remark 1.8. Since in [1] they just considered sequences with isolated sub-limits, or a little complex case with only one limit point, this ambiguous treatment of essential subsequences didn't lead to serious mistakes.

The following theorem is the most important result of [1], which will be cited and reviewed later.

Theorem 1.9 (Theorem 4, [1]). *Suppose $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and $S(x) = \{a_1, a_2, \ldots, a_m\}$ is a finite set, where $a_i \neq a_j$ if $i \neq j$. Then*

$$\sum_{0<a_j\in S(x)} a_j w_l(a_j) + \sum_{0>a_j\in S(x)} a_j w^u(a_j) \le L(x)$$
$$\le \sum_{0<a_j\in S(x)} a_j w^u(a_j) + \sum_{0>a_j\in S(x)} a_j w_l(a_j).$$

If $w(a_j)$ exists for each j, then x is almost convergent and for any Banach limit L, $L(x) = \sum_{j=1}^{m} a_j w(a_j)$.

This form of $L(x) = \sum_{j=1}^{m} a_j w(a_j)$ is much like the *integration sum* in measure and integration theory, so we ask the question whether the unique Banach limit value of almost convergent sequence could be expressed as an integral form? Previous work shows this is related to the distribution of values appearing in the sequence. In [5], the concept of *uniform distribution of sequences* was introduced

[1]Special thanks goes to Prof. J. L. Li for discussion with him on this proposition. In fact, it was him that first pointed out this proposition and provided a proof for the sufficient condition.

as following: Suppose $x \in l^\infty$ is a $[0,1]$-valued sequence, i.e. $0 \le x(n) \le 1$ for each $n \in \mathbb{N}$. x is called *uniformly distributed* if for any $[a,b) \subseteq [0,1]$,

$$\lim_{N\to\infty} \frac{A(\{n \in \mathbb{N} : x(n) \in [a,b), n \le N\})}{N} = b - a.$$

Now we want to generalize the concept of distribution to cover both the uniform and ununiform cases.

2. Main results

Definition 2.1. A sequence $x := \{x(n)\}_{n=1}^\infty \in l^\infty$ is called properly distributed if for any Borel subset S of $[-\|x\|_\infty, \|x\|_\infty]$ it holds that

$$\begin{aligned} w(x,S) &= \liminf_{n\to\infty} \frac{A(\{k : x(k+i) \in S, k = 0,1,\dots,n-1\})}{n} \\ &= \limsup_{n\to\infty} \frac{A(\{k : x(k+i) \in S, k = 0,1,\dots,n-1\})}{n} \end{aligned}$$

exists uniformly in $i \in \mathbb{N}$ and $w(x,S)$ is called the weight of x with respect to S.

If we treat a properly distributed sequence x as a function defined on $\mathbb{N}$, x is analogous to the measurable function in real analysis, with $w(x,S)$ corresponding to some measure $\mu(\{n : x(n) \in S\})$ over $\mathbb{N}$. Though $w(x,S)$ indeed has some similar behavior as a measure like nonnegativity and finite additivity, $w(x,S)$ is not a measure in general setting, for it fails to satisfy countable additivity. Here is an illustrating example:

Example 2.2. Let $s_1 = \{\underbrace{1,\dots,1}_{n-times}, \underbrace{0,0,\dots}_{otherwise}\}$, which is obviously properly distributed. If there exists a measure μ over $\mathbb{N}$ such that $\mu(\{n : x(n) \in S\}) = w(x,S)$ for any properly distributed sequence $x \in l^\infty$ and Borel subset S, then $\mu(\{1,2,\dots,n\}) = w(x,[1-\varepsilon,1+\varepsilon)) = 0$, where ε is a sufficiently small positive number. Similarly, it can further be implied that for any finite subset E of $\mathbb{N}$ it always holds $\mu(E) = 0$. Since μ is countably additive and $\mathbb{N}$ is the union of pairwise disjoint finite subsets, it follows that $\mu(\mathbb{N}) = 0$. However, if we set $s_2 = \{1,\dots,1,\dots\}$, then s_2 is properly distributed and $\mu(\mathbb{N}) = w(s_2,[1-\varepsilon,1+\varepsilon)) = 1$, which leads to a contradiction. Thus, such measure μ over $\mathbb{N}$ doesn't exist.

From Example 2.2, you may have already realized that s_1 and s_2 represent a simple but useful class of properly distributed sequences. Hence, we naturally give the following definition of *simply distributed sequences*, which would play the similar role as "simple functions" in real analysis.

Definition 2.3. A sequence $s := \{s(n)\}_{n=1}^\infty \in l^\infty$ is called simply distributed if s is finitely-valued with range $\{a_1,\dots,a_m\}$ and it holds that

$$\begin{aligned} w(s,a_j) &= \liminf_{n\to\infty} \frac{A(\{k : s(k+i) = a_j, k =, 0,1,\dots,n-1\})}{n} \\ &= \limsup_{n\to\infty} \frac{A(\{k : s(k+i) = a_j, k = 0,1,\dots,n-1\})}{n} \end{aligned}$$

exists uniformly in $i \in \mathbb{N}$ for $j = 1,\dots,m$ and $w(s,a_j)$ is called the weight of s with respect to a_j.

Though we cannot bring our work into the framework of measure and integration(In fact, we really tried to do so at the beginning of our research.), we still find much common feature between them, which suggests us to generalize the measure-integration procedure in real analysis to obtain a *formal* integral to express the unique Banach limit of almost convergent sequence. This would partially answer the open question of [2].

Theorem 2.4. *If $s \in l^\infty$ is a simply distributed sequence with finite range $\{a_1, \ldots, a_m\}$, then it is almost convergent with the unique Banach limit $L(s) = \sum_{j=1}^m a_j w(s, a_j)$.*

Proof. Let $S(s)$ denote the set of all sub-limits of s. Since s is finitely-valued, we have $S(s) \subseteq \{a_1, \ldots, a_m\}$ is finite. Moreover, if $a_j \notin S(s)$, then a_j must appear finite times in s with $w(s, a_j) = 0$. Hence, by Theorem 4 of [1], it implies that s is almost convergent and for any Banach limit L, $L(s) = \sum_{j=1}^m a_j w(s, a_j)$. □

From Theorem 2.4, we can see that for any simply distributed sequence s, its unique Banach limit could be expressed as formal integral $L(s) = \sum_{j=1}^m a_j w(s, a_j)$. Then it naturally arises the question that whether it is still true for general properly distributed sequences. To this end, we'd like to generalize the procedure of integration in real analysis. Firstly, let us approximate properly distributed sequences by simply distributed sequences.

Lemma 2.5. *For any properly distributed element $x \in l^\infty$, there is a sequence of simply distributed elements $\{s_k\}_{k=1}^\infty \subseteq l^\infty$ such that $\lim_{k\to\infty} s_k = x$ under the norm $\|\cdot\|_\infty$ in l^∞.*

Proof. For $k \in \mathbb{N}$, there is a partition

$$T_k : -\|x\|_\infty = a_0 < \ldots < a_{m_k} = \|x\|_\infty$$

of $[-\|x\|_\infty, \|x\|_\infty]$ such that $\|T_k\| < 1/k$. Define

$$s_k(n) = \begin{cases} a_0, & \text{if } a_0 \le x(n) < a_1, \\ \cdots & \cdots, \\ a_{m_k-1}, & \text{if } a_{m_k-1} \le x(n) < a_{m_k}. \end{cases} \qquad n = 1, 2, 3, \ldots$$

Since x is properly distributed, it follows easily that each s_k is simply distributed. According to the above construction, it is obvious that $\|s_k - x\|_\infty < 1/k$. Thus $\lim_{k\to\infty} s_k = x$. □

Theorem 2.6. *If $x \in l^\infty$ is any properly distributed sequence, then x is almost convergent. And if $\{s_k\}_{k=1}^\infty$ is any sequence of simply distributed sequences convergent to x under the $\|\cdot\|_\infty$ norm, for any Banach limit L, it always holds that $\lim_{k\to\infty} L(s_k) = L(x)$.*

Proof. For any Banach limit L, since L is a bounded linear functional on l^∞ and $\lim_{k\to\infty} s_k = x$, it follows that $\lim_{k\to\infty} L(s_k) = L(x)$. By Theorem 2.4, the value of each $L(s_k)$ is independent of L, thus so is $L(x)$. We conclude that x is almost convergent and the unique Banach limit is $\lim_{k\to\infty} L(s_k)$. □

Now we want to use the new theory to review the work of [1], which will be shown to be a special case in our framework.

Lemma 2.7. *Let $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$. Suppose a is an isolated sub-limit of x, and there exists $\varepsilon_0 > 0$ such that $(a-\varepsilon_0, a+\varepsilon_0) \bigcap S(x) = \{a\}$. Then for any $0 < \varepsilon \leq \varepsilon_0$, $w(x, [a-\varepsilon, a+\varepsilon))$ exists if and only if $w(a)$ does. Moreover, if they both exist, they are equal.*

Proof. Like Proposition 1.7, for any $0 < \varepsilon \leq \varepsilon_0$, let $\{x(n_k)\}$ denote all the terms of x that lying in $[a-\varepsilon, a+\varepsilon)$. Then, similarly, it is easy to show that $\{x(n_k)\}$ is an essential subsequence of a. And, for any $n, i \in \mathbb{N}$, we have

$$\begin{aligned}&\frac{A(\{j : a-\varepsilon \leq x(i+j) < a+\varepsilon, j = 0, 1, \ldots, n-1\})}{n}\\ =&\frac{A(\{k : i \leq n_k \leq i+n-1\})}{n}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\limsup_{n\to\infty} \frac{A(\{j : a-\varepsilon \leq x(i+j) < a+\varepsilon, j = 0, 1, \ldots, n-1\})}{n}\\ =&\limsup_{n\to\infty} \frac{A(\{k : i \leq n_k \leq i+n-1\})}{n},\end{aligned}$$

and

$$\begin{aligned}&\liminf_{n\to\infty} \frac{A(\{j : a-\varepsilon \leq x(i+j) < a+\varepsilon, j = 0, 1, \ldots, n-1\})}{n}\\ =&\liminf_{n\to\infty} \frac{A(\{k : i \leq n_k \leq i+n-1\})}{n}.\end{aligned}$$

Now it is clear that $w(x, [a-\varepsilon, a+\varepsilon))$ exists if and only if $w(a)$ does. And, if they both exist, they are equal. □

Now it's time to include Theorem 4([1]) into our framework.

Theorem 2.8. *Suppose $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and $S(x) = \{a_1, a_2, \ldots, a_m\}$ is a finite set, where $a_i \neq a_j$ if $i \neq j$. If $w(a_j)$ exists for each j, then x is properly distributed.*

Proof. For any interval $[c, d)$, if $[c, d) \bigcap \{a_1, a_2, \ldots, a_m\} = \emptyset$, there would be at most finite terms in $[c, d)$, so

$$\begin{aligned}w(x, [c, d)) &= \liminf_{n\to\infty} \frac{A(\{k : x(k+i) \in [c, d), k = 0, 1, \ldots, n-1\})}{n}\\ &= \limsup_{n\to\infty} \frac{A(\{k : x(k+i) \in [c, d), k = 0, 1, \ldots, n-1\})}{n}\\ &= 0\end{aligned}$$

exists uniformly in $i \in \mathbb{N}$. Otherwise, there are some a_js in $[c, d)$. Without loss of generality, we can assume only a_j lying $[c, d)$. In fact, if there are more than one such a_j, we can decompose $[c, d)$ into disjoint subintervals such that each contains only one a_j. From Lemma 2.7, since $w(a_j)$ exists, we also have $w(x, [c, d))$ exists, and $w(x, [c, d)) = w(a_j)$. Thus we have proved that x is properly distributed. □

Moreover, we can reobtain the unique Banach limit of x above, using the approximation method by simply distributed sequences.

Corollary 2.9. Suppose $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and $S(x) = \{a_1, a_2, \dots, a_m\}$ is a finite set, where $a_i \neq a_j$ if $i \neq j$. If $w(a_j)$ exists for each j, then x is almost convergent, with the unique Banach limit $L(x) = \sum_{j=1}^{m} a_j w(a_j)$ for any Banach limit L.

Proof. For any sufficiently big $k \in \mathbb{N}$, define

$$s_k(n) = \begin{cases} a_j, & \text{if } a_j - 1/k \le x(n) < a_j + 1/k, \\ x(n), & \text{otherwise.} \end{cases} \quad j = 1, \dots, m; n \in \mathbb{N}.$$

It is easy to see that each s_k is a simply distributed sequence with only $w(s_k, a_j) \neq 0$, and $L(s_k) = \sum_{j=1}^{m} a_j w(s_k, [a_j - 1/k, a_j + 1/k)) = \sum_{j=1}^{m} a_j w(a_j)$. From the construction of $\{s_k\}_{k=1}^{\infty}$, $\lim_{k\to\infty} s_k = x$ under the $\|\cdot\|_{\infty}$ norm. Then it follows that $L(x) = \lim_{k\to\infty} L(s_k) = \sum_{j=1}^{m} a_j w(a_j)$. □

In Theorem 5 and 6 of [1], sequences whose sub-limit sets have limit points are considered. In order to keep the form $L(x) = \sum_{a\in S(x)} aw(a)$, the authors made a great effort to give a complex definition for the weight of limit points of $S(x)$. Now, from our distribution viewpoint, it is very easy to understand those complex formulae. Let us take Theorem 5 [1] for example, Theorem 6 [1] is treated in a similar way locally at each limit point of $S(x)$.

Theorem 2.10. *Suppose $x := \{x(n)\}_{n=1}^{\infty} \in l^{\infty}$ and $S(x)$ is infinite but countable and has a unique limit point p, that is $S'(x) = \{p\}$. If, furthermore, $w(a)$ exists for all $a \in S(x)$ and $a \neq p$, then x is properly distributed, and for any Banach limit L, $L(x) = \sum_{a\in S(x)} aw(a)$, where $w(p) = 1 - \sum_{p\neq a\in S(x)} w(a)$.*

Proof. For any sufficiently big $k \in \mathbb{N}$, define

$$s_k(n) = \begin{cases} p, & \text{if } p - 1/k \le x(n) \le p + 1/k, \\ a_j, & \text{if } a_j - 1/k \le x(n) < a_j + 1/k, \text{ and } a_j \notin [p - 1/k, p + 1/k), \\ x(n), & \text{otherwise.} \end{cases}$$

Since there are only finite $a_j \notin [p-1/k, p+1/k)$, each s_k is properly distributed and $\lim_{k\to\infty} s_k = x$. Moreover, from Lemma 2.7, we have

$$L(s_k) = \sum_{a_j \notin [p-1/k, p+1/k)} a_j w(a_j) + p(1 - \sum_{a_j \notin [p-1/k, p+1/k)} w(a_j)).$$

Let $k \to \infty$, it follows that $L(x) = \lim_{k\to\infty} L(s_k) = \sum_{a\in S(x)} aw(a)$, where $w(p) = 1 - \sum_{p\neq a\in S(x)} w(a)$. □

References

1. Feng, B. Q. and Li, J. L., *Some estimations of Banach limits* J. Math. Anal. Appl. 323(2006) No. 1, 481-496. MR2262220 46B45 (46A45).
2. Feng, B. Q., *Two almost convergent sequences which were equipped an uncountable set of limit points*, preprint.
3. Lorentz, G. G., *A contribution to the theory of divergent sequences*, Acta Math., 80(1948) 167-190. MR0027868 (10,367e).
4. Sucheston, L., *Banach limits*, Amer. Math. Monthly, 74 (1967) 308-311. MR0225145 (37#740).
5. Kuipers, L. and Niederreiter, H., *Uniform distribution of sequences*, New York, Wiley, 1974.

Department of Mathematics & The Academy of Fundamental and Interdisciplinary Science, Harbin Institute of Technology, Harbin 150001, Heilongjiang, People's Republic of China
E-mail address: hityou1982@gmail.com

Department of Mathematical Sciences, Kent State University, Tuscarawas, New Philadelphia, OH 44663, USA
E-mail address: bfeng@kent.edu