Geometric Means and Hadamard Products

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Abstract. Ando [1] proved that for $m$ commuting positive definite matrices, the $m$-fold Hadamard product of their geometric mean is bounded above by their Hadamard product. We obtain a natural extension to the non-commutative case.

In what follows, if $M$ is a positive definite $n \times n$ matrix and $\alpha \in \mathbb{R}$, then $M^\alpha$ will denote its unique positive $\alpha^{th}$ power. In 1979, Ando [1] developed a robust definition of the geometric mean of two positive definite $n \times n$ matrices $M_1$ and $M_2$ that had been introduced by Pusz and Woronowicz [5]:

$$G(M_1, M_2) := M_2^{1/2} \left( M_2^{-1/2} M_1 M_2^{-1/2} \right)^{1/2} M_2^{1/2}.$$

The geometric mean is symmetric in $M_1$, $M_2$, monotone in each variable and satisfies the arithmetic-geometric mean inequality

$$G(M_1, M_2) \leq A(M_1, M_2) := \frac{1}{2} (M_1 + M_2).$$

Given $0 < \alpha < 1$, the geometric mean is generalized to the $\alpha$-mean $G^{(\alpha)}(M_1, M_2)$:

$$G^{(\alpha)}(M_1, M_2) := M_2^{1/2} (M_2^{-1/2} M_1 M_2^{-1/2})^{\alpha} M_2^{1/2}.$$

The usual geometric mean is just $G^{(1/2)}(M_1, M_2)$. In stead of the arithmetic-geometric mean inequality the Young inequality holds for $\alpha$-mean:

$$G^{(\alpha)}(M_1, M_2) \leq A^{(\alpha)}(M_1, M_2) := \alpha M_1 + (1 - \alpha) M_2.$$

Ando [1] also researched the interaction of this geometric mean with the Hadamard (or Schur) product. If $M = (m_{ij})$, $N = (n_{ij})$ are matrices of the same size, their Hadamard product $M \circ N$ is the matrix of entry-wise products:

$$M \circ N := (m_{ij} n_{ij}).$$

Ando [1, Theorem 13] proved that for positive definite $n \times n$ matrices $M$, $N$ we have

$$G(M, N) \circ G(M, N) \leq M \circ N, \quad (A_1)$$

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which, in the commutative case, reduces to
\[(MN)^{\frac{1}{2}} \circ (MN)^{\frac{1}{2}} \leq M \circ N.\]

Using a different method, he succeeded in generalizing his inequality to the case of several commuting positive definite \(n \times n\) matrices [1, Theorem 12]:
\[
\prod_{i=1}^{m} \circ \left( \prod_{i=1}^{m} M_i \right)^{\frac{1}{m}} \leq \prod_{i=1}^{m} \circ M_i. \tag{A_2}
\]

However, at the time, the notion of geometric mean for several non-commuting matrices was not available; therefore, inequality \((A_1)\) was not developed beyond the case of two matrices.

In 1994, M. Sagae and K. Tanabe [6] successfully developed an approach to the geometric mean — in fact, the weighted geometric mean — of several positive definite matrices. The main purpose of this paper is to blend the ideas of Ando and the concepts of Sagae and Tanabe to extend inequality \((A_1)\) to the case of several positive definite \(n \times n\) matrices. As a result, we are able to provide a simpler proof of inequality \((A_2)\).

The definition \(G_{w} (M_1, M_2, \ldots, M_m)\) of the geometric mean of an \(m\)-tuple of positive definite matrices \((M_1, M_2, \ldots, M_m)\) \((m > 2)\) was given by Sagae and Tanabe in 1994.

When \(m = 2\), define
\[G_{w} (M_1, M_2) := G^{(w_1)} (M_1, M_2).\]

Suppose that the definition for the case \(m - 1\) has been well established. Now given an \(m\)-tuple of positive numbers \((w_1, w_2, \ldots, w_m)\) summing to 1, define
\[G_{w} (M_1, M_2, \ldots, M_m) := G^{(\sum_{j=1}^{m-1} w_j)} (G_{\tilde{w}} (M_1, M_2, \ldots, M_{m-1}), M_m)\]
where
\[
\tilde{w} = \left( w_1 / \sum_{j=1}^{m-1} w_j, w_2 / \sum_{j=1}^{m-1} w_j, \ldots, w_{m-1} / \sum_{j=1}^{m-1} w_j \right).
\]

In order to work effectively with the general geometric mean of Sagae and Tanabe, it is convenient to introduce the \((\alpha_1, \ldots, \alpha_k)\)-power mean for \((k + 1)\)-tuple positive definite matrices. Suppose that \(M_i\) \((i \geq 1)\) are positive definite \(n \times n\) matrices and \(\alpha_i\) \((i \geq 1)\) are real scalars. Starting with the two matrix basis, we can continue recursively to define
\[G^{(\alpha_1, \ldots, \alpha_k)} (M_1, \ldots, M_{k+1}) = G^{(\alpha_k)} (G^{(\alpha_1, \ldots, \alpha_{k-1})} (M_1, \ldots, M_k), M_{k+1})\]
for \(k \geq 2\).

When \(\alpha_i = 1 - \left( w_{i+1} / \sum_{j=1}^{i+1} w_j \right)\) for \(i = 1, \ldots, m - 1\), we have
\[G^{(\alpha_1, \ldots, \alpha_{m-1})} (M_1, \ldots, M_m) = G_{w} (M_1, \ldots, M_m).\]
This general definition of geometric mean has many good properties, but in the case of equal weights it is not symmetric for \( k > 2 \) (see [3]). For us, it will be significant that the weighted geometric mean satisfies an arithmetic-geometric mean inequality.

**Theorem** ([6], Theorem 1) *Let \( w \) be an \( m \)-tuple of positive numbers \( (w_1, w_2, \ldots, w_m) \) summing to 1 and let \( M_i \) \((1 \leq i \leq m)\) be positive definite \( n \times n \) matrices. Then
\[
G_w(M_1, \ldots, M_k) \leq A_w(M_1, \ldots, M_k) := w_1M_1 + \cdots + w_kM_k.
\]
The inequality is strict unless \( M_1 = \cdots = M_k \)."

From inequality \((A_1)\) it is natural to conjecture that when \( M_i \) \((i = 1, 2, 3)\) are positive definite matrices (possibly non-commuting), the inequality
\[
G_w(M_1, M_2, M_3) \circ G_w(M_1, M_2, M_3) \circ G_w(M_1, M_2, M_3) \leq M_1 \circ M_2 \circ M_3
\]
holds true. The answer is positive, and in fact the order of the matrices is unimportant. This is to some extent surprising, since Ando’s proof of inequality \((A_1)\) depends on the symmetry of the geometric mean of two matrices. The main result in this paper is:

**Theorem 1** *Let \( w \) be an \( m \)-tuple of positive numbers \( (w_1, w_2, \ldots, w_m) \) summing to 1 and let \( M_i \) \((1 \leq i \leq m)\) be positive definite \( n \times n \) matrices. If \((i_1, i_2, \ldots, i_m), (j_1, j_2, \ldots, j_m), \ldots, (k_1, k_2, \ldots, k_m)\) are arbitrary \( m \)-permutations of \( \{1, 2, \ldots, m\} \), then
\[
G_w(M_{i_1}, \ldots, M_{i_m}) \circ G_w(M_{j_1}, \ldots, M_{j_m}) \circ \cdots \circ G_w(M_{k_1}, \ldots, M_{k_m}) \leq \prod_{i=1}^{m} \circ M_i.
\]

In order to obtain the proof of Theorem 1, we need to develop some properties of tensor products of matrices. If \( M = (m_{ij}) \) is an \( k \times l \) matrix and \( N = (n_{ij}) \) is an \( s \times t \) matrix, then their tensor (or Kronecker) product is the \( ks \times lt \) matrix
\[
M \otimes N := \begin{bmatrix}
m_{11}N & \cdots & m_{1l}N \\
\vdots & \ddots & \vdots \\
m_{k1}N & \cdots & m_{kl}N
\end{bmatrix}.
\]
The tensor product of finitely many matrices can be defined by induction.

The basic properties of the tensor product can be found in [2, p. 15] and [1, p. 224]. We need two more properties that we were unable to find in the literature.

**Proposition 1** *(i)* *Let \( M_i \) \((1 \leq i \leq k)\) be \( m \times m \) matrices and let \( N_i \) \((1 \leq i \leq k)\) be \( n \times n \) matrices. Then
\[
\prod_{i=1}^{k} (M_i \otimes N_i) = \left( \prod_{i=1}^{k} M_i \right) \otimes \left( \prod_{i=1}^{k} N_i \right).
\]

*(ii)* *Let \( M_i \) be positive definite \( n_i \times n_i \) matrices \((1 \leq i \leq k)\). Then, for any real number \( \alpha \),
\[
\left( \prod_{i=1}^{k} \otimes M_i \right)^{\alpha} = \prod_{i=1}^{k} \otimes M_i^{\alpha}.
\]
Proof  (i) This follows easily by induction from the observation that 

\[(A \otimes B)(C \otimes D) = (AC) \otimes (BD),\]

which can be verified immediately from the definition of the tensor product of matrices.

(ii) Again, this follows by induction from the case of the product of two matrices. Since \(M_1\) and \(M_2\) are positive definite, we can write (see [4, Theorem 2.5.4]) \(M_1 = U_1^* S U_1\) and \(M_2 = U_2^* T U_2\), using appropriate unitary matrices \(U_1\) and \(U_2\), and positive definite diagonal matrices \(S\) and \(T\).

By (i),

\[M_1 \otimes M_2 = (U_1^* \otimes U_2^*)(S \otimes T)(U_1 \otimes U_2) = (U_1 \otimes U_2)^*(S \otimes T)(U_1 \otimes U_2)\]

and, since \(U_1 \otimes U_2\) is unitary, the functional calculus (see [1, p. 212]) allows us to write

\[M_1 \otimes M_2 = (U_1^* \otimes U_2^*)(S \otimes T)(U_1 \otimes U_2) = (U_1^* S U_1)^\alpha \otimes (U_2^* T U_2)^\alpha = M_1^\alpha \otimes M_2^\alpha.\]

In [1], Ando pointed out a fundamental commutativity relation between the geometric mean and tensor product of two positive definite matrices, namely

\[G(M_1 \otimes M_2, N_1 \otimes N_2) = G(M_1, N_1) \otimes G(M_2, N_2).\]  

(1)

In fact, this can readily be extended to the \(\alpha\)-power mean. The analog of identity (1) is

\[G^{(\alpha)}(M_1 \otimes M_2, N_1 \otimes N_2) = G^{(\alpha)}(M_1, N_1) \otimes G^{(\alpha)}(M_2, N_2),\]  

(2)

and this follows easily from Proposition 1.

A multi-stage induction argument leads to a simple, but powerful extension of identity (2):

**Proposition 2** Let \(M_{ij}\) (\(1 \leq i \leq m, 1 \leq j \leq k\)) be positive definite \(n \times n\) matrices, and let \(w\) be an \(m\)-tuple of positive numbers \((w_1, w_2, \ldots, w_m)\) summing to 1. Then

\[G_w \left( \prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj} \right) = \prod_{j=1}^{k} \otimes G_w(M_{1j}, \ldots, M_{mj}).\]

**Proof** With the help of Proposition 1, a preliminary induction shows that

\[G^{(\alpha_1, \ldots, \alpha_{m-1})}(M_1 \otimes N_1, \ldots, M_m \otimes N_m) = G^{(\alpha_1, \ldots, \alpha_{m-1})}(M_1, \ldots, M_m) \otimes G^{(\alpha_1, \ldots, \alpha_{m-1})}(N_1, \ldots, N_m),\]

and then a second induction gives

\[G^{(\alpha_1, \ldots, \alpha_{m-1})} \left( \prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj} \right) = \prod_{j=1}^{k} \otimes G^{(\alpha_1, \ldots, \alpha_{m-1})}(M_{1j}, \ldots, M_{mj}).\]


Setting $\alpha_i = 1 - \left( w_{i+1} / \sum_{j=1}^{i+1} w_j \right)$ for $i = 1, \ldots, m - 1$ yields the statement of Proposition 2. We omit the simple details.

In many situations, properties of tensor products transfer to Hadamard products. This is thanks to an important connection between the two products (e.g. see [1, Lemma 4]): there is a positive linear map $\Phi_k$ from $n^k$-dimensional space to $n$-dimensional space of matrices such that, for all $n \times n$ matrices $M_i$ ($1 \leq i \leq k$),

$$\Phi_k \left( \prod_{i=1}^{k} \otimes M_i \right) = \prod_{i=1}^{k} \circ M_i. \quad (3)$$

With this, we can quickly identify the key to our proof of Theorem 1:

**Proposition 3** Let $M_{ij}$ ($1 \leq i \leq m, 1 \leq j \leq k$) be positive definite $n \times n$ matrices, and let $w$ be an $m$-tuple of positive numbers $(w_1, w_2, \ldots, w_m)$ summing to 1. Then

$$\prod_{j=1}^{k} \otimes G_w(M_{1j}, \ldots, M_{mj}) \leq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes M_{ij}$$

and

$$\prod_{j=1}^{k} \circ G_w(M_{1j}, \ldots, M_{mj}) \leq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \circ M_{ij}.$$

**Proof** By Sagae and Tanabe’s arithmetic-geometric mean inequality and Proposition 2,

$$\prod_{j=1}^{k} \otimes G_w(M_{1j}, \ldots, M_{mj}) = G_w \left( \prod_{j=1}^{k} \otimes M_{1j}, \ldots, \prod_{j=1}^{k} \otimes M_{mj} \right) \leq \sum_{i=1}^{m} w_i \prod_{j=1}^{k} \otimes M_{ij}.$$  

The Hadamard product inequality follows by (3) from the tensor product inequality. \qed

**Proof of Theorem 1** In Proposition 3, let

$$(M_{11}, \ldots, M_{m1}) = (M_{i1}, \ldots, M_{im}),$$

$$(M_{12}, \ldots, M_{m2}) = (M_{j1}, \ldots, M_{jm}),$$

$$\vdots$$

$$(M_{1m}, M_{2m}, \ldots, M_{mm}) = (M_{k1}, \ldots, M_{km}).$$

Then, since the Hadamard product is commutative,

$$G_w(M_{i1}, \ldots, M_{im}) \circ \cdots \circ G_w(M_{k1}, \ldots, M_{km})$$

$$= \prod_{j=1}^{m} \circ G_w(M_{1j}, \ldots, M_{mj}) \leq \sum_{i=1}^{m} w_i \prod_{j=1}^{m} \circ M_{ij} = \sum_{i=1}^{m} w_i \prod_{j=1}^{m} \circ M_{j} = \prod_{i=1}^{m} \circ M_i. \quad \square$$

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It is clear that inequality $(A_2)$ is an immediate corollary of Theorem 1. Let $(i_1, i_2, \ldots, i_m), (j_1, j_2, \ldots, j_m), \ldots, (k_1, k_2, \ldots, k_m)$ be arbitrary $m$-permutations of $\{1, 2, \ldots, m\}$. If $M_i$ ($1 \leq i \leq m$) commute and $w_1 = w_2 = \cdots = w_m = 1/m$, then

$$G_w(M_{i_1}, \ldots, M_{i_m}) = G_w(M_{j_1}, \ldots, M_{j_m}) = \cdots = G_w(M_{k_1}, \ldots, M_{k_m}) = \left(\prod_{i=1}^{m} M_i\right)^{1/m}.$$ 

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