Definitions:

- **One-to-one function:** is a function in which no two elements of the domain \( A \) have the same image. In other words, \( f \) is a one-to-one function if \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \).

- **Inverse function:** Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). Then its inverse function, denoted \( f^{-1} \), has domain \( B \) and range \( A \) and is defined by

\[
f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y
\]
for any \( y \) in \( B \).

Finding the inverse of a one-to-one function:

1. Replace \( f(x) \) with \( y \).
2. Interchange \( x \) and \( y \).
3. Solve this equation for \( y \). The resulting equation is \( f^{-1}(x) \).

Important Properties:

- **Horizontal line test:** A function is one-to-one if no horizontal line intersects its graph more than once.

- **Property of inverse functions:** Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). The inverse function \( f^{-1} \) satisfies

\[
f^{-1}(f(x)) = x \quad \text{for every} \quad x \in A
\]

\[
\text{and} \quad f(f^{-1}(x)) = x \quad \text{for every} \quad x \in B
\]

- The inverse of \( f^{-1} \) is \( f \). So, we say that \( f \) and \( f^{-1} \) are inverses of each other.

- The inverse function interchanges the domain and range. Namely,

\[
\text{Domain of } f = \text{Range of } f^{-1}
\]

\[
\text{Range of } f = \text{Domain of } f^{-1}
\]

- The graph of \( f^{-1} \) is found by reflecting the graph of \( f \) across the line \( y = x \).

- Only a one-to-one function can have an inverse.

Common Mistakes to Avoid:

- The \(-1\) in the inverse \( f^{-1} \) is NOT an exponent. Be aware that

\[
f^{-1}(x) \neq \frac{1}{f(x)}.
\]

- In order for \( f \) to be a one-to-one function it must first be a function. Therefore, in order for \( f \) to be a one-to-one function it must pass both the vertical and horizontal line tests.
PROBLEMS

1. Determine whether each function is a one-to-one function. (Remember \( f \) is one-to-one if \( f(x_1) = f(x_2) \) implies that \( x_1 = x_2 \).)

   (a) \( f(x) = 8x - 3 \)

   \[ f(x_1) = f(x_2) \]
   \[ 8x_1 - 3 = 8x_2 - 3 \]
   \[ 8x_1 = 8x_2 \]
   \[ x_1 = x_2 \]

   \[ f \text{ is a one-to-one function} \]

   (b) \( f(x) = x^4 + 7 \)

   \[ f(x_1) = f(x_2) \]
   \[ x_1^4 + 7 = x_2^4 + 7 \]
   \[ x_1^4 = x_2^4 \]
   \[ \sqrt[4]{x_1^4} = \sqrt[4]{x_2^4} \]
   \[ x_1 = \pm x_2 \]

   \[ f \text{ is NOT a one-to-one function} \]

2. If \( f \) is a one-to-one function for which \( f(1) = 7 \), \( f(-3) = 9 \) and \( f(6) = 2 \) find \( f^{-1}(9) \), \( f^{-1}(7) \) and \( f^{-1}(2) \).

   Since \( f \) is a one-to-one function we know that it has an inverse. Remember that the inverse interchanges the \( x \) and \( y \) variable. Therefore,

   \[ f^{-1}(9) = -3, \quad f^{-1}(7) = 1, \quad f^{-1}(2) = 6 \]

3. Find the inverse of \( f \).

   (a) \( f(x) = 3x - 5 \)

   \[ f(x) = 3x - 5 \]
   \[ y = 3x - 5 \]
   \[ x = 3y - 5 \]
   \[ x + 5 = 3y \]
   \[ \frac{x + 5}{3} = y \]

   \[ f^{-1}(x) = \frac{x + 5}{3} \]

   (b) \( f(x) = 9 - 4x \)

   \[ f(x) = 9 - 4x \]
   \[ y = 9 - 4x \]
   \[ x = 9 - 4y \]
   \[ 4y = 9 - x \]
   \[ y = \frac{9 - x}{4} \]

   \[ f^{-1}(x) = \frac{9 - x}{4} \]

   (c) \( f(x) = \frac{x - 2}{6} \)

   \[ f(x) = \frac{x - 2}{6} \]
   \[ y = \frac{x - 2}{6} \]
   \[ x = \frac{y - 2}{6} \]
   \[ 6x = y - 2 \]
   \[ 6x + 2 = y \]

   \[ f^{-1}(x) = 6x + 2 \]
(d) \( f(x) = \frac{2}{x - 4} \)

\[
\begin{align*}
  f(x) &= \frac{2}{x - 4} \\
  y &= \frac{2}{x - 4} \\
  x &= \frac{2}{y - 4} \\
  x(y - 4) &= 2 \\
  xy - 4x &= 2 \\
  xy &= 4x + 2 \\
  y &= \frac{4x + 2}{x} \\
  f^{-1}(x) &= \frac{4x + 2}{x}
\end{align*}
\]

Note that with the restriction \( x \leq 0 \), the function \( f(x) = x^2 \) becomes a one-to-one function.

\[
\begin{align*}
  f(x) &= x^2 \\
  y &= x^2 \\
  x &= y^2 \\
  \sqrt{x} &= \sqrt{y^2} \\
  \pm \sqrt{x} &= y
\end{align*}
\]

Now we need to decide whether our answer is \( \sqrt{x} \) or \(-\sqrt{x}\). Remember that the range of \( f^{-1} \) is the domain of \( f \). Since the domain of \( f \) is \( x \leq 0 \) (negative numbers and zero), we need to choose \(-\sqrt{x}\).

\[
\begin{align*}
  f^{-1}(x) &= -\sqrt{x}
\end{align*}
\]

(g) \( f(x) = \sqrt{4x - 7} \)

\[
\begin{align*}
  f(x) &= \sqrt{4x - 7} \\
  y &= \sqrt{4x - 7} \\
  x &= \sqrt{4y - 7} \\
  x^2 &= (\sqrt{4y - 7})^2 \\
  x^2 &= 4y - 7 \\
  x^2 + 7 &= 4y \\
  \frac{x^2 + 7}{4} &= y
\end{align*}
\]

\[
\begin{align*}
  f^{-1}(x) &= \frac{x^2 + 7}{4}, \quad x \geq 0
\end{align*}
\]
4. Given the graph of $f$, sketch the graph of $f^{-1}$.

To do this remember that the graph of $f^{-1}$ is the reflection of $f$ across the line $y = x$. Also, $f^{-1}$ interchanges the $x$ and $y$ variables. Therefore, we will interchange the $x$– and $y$–coordinates of each ordered pair. Once we graph these we will connect them with straight lines.