Definition:

- **Composition function**: Given two functions f and g, the composition function $f \circ g$ is defined by

$$\boxed{(f \circ g)(x) = f(g(x)).}$$

  In other words, given a number x, we first apply g to it and then we apply f to the result. Here, f is the outside function and g is the inside function.

Important Properties:

- Let c be any constant. There are two ways to find $(f \circ g)(c)$. You could first evaluate $g(c)$ and then evaluate f at the result. Or you could first find $(f \circ g)(x)$ and then evaluate the resulting function at c.
- To find $(f \circ g)(x)$, remember to substitute the value $g(x)$ into every variable that occurs in f.
- The order of the functions is important. In general,

$$(f \circ g)(x) \neq (g \circ f)(x).$$

- Other composition functions are defined similarly. Namely,

$$(g \circ f)(x) = g(f(x))$$
$$(f \circ f)(x) = f(f(x))$$
$$(g \circ g)(x) = g(g(x))$$

Common Mistakes to Avoid:

- Composition of functions is different than the multiplication of functions. Therefore,

$$(f \circ g)(x) \neq f(x) \cdot g(x).$$

PROBLEMS

1. **If** $f(x) = 3x - 5$ **and** $g(x) = x + 2$ **find** $(f \circ g)(x)$ **and** $(g \circ f)(x)$.

To find $(f \circ g)(x)$ we will substitute g in for every variable that occurs in f.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x+2) \\ &= 3(x+2) - 5 \\ &= 3x + 6 - 5 \\ &= 3x + 1\end{aligned}$$

To find $(g \circ f)(x)$ we will substitute f into every variable that occurs in g.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x-5) \\ &= (3x-5) + 2 \\ &= 3x - 3\end{aligned}$$

$$\boxed{(f \circ g)(x) = 3x + 1}$$

$$\boxed{(g \circ f)(x) = 3x - 3}$$

2. **Given** $f(x) = x^2 - 5x + 1$ **and** $g(x) = 2x + 1$**, find** $(f \circ g)(x)$ **and** $(g \circ f)(x)$.

To find $(f \circ g)(x)$ we will substitute g in for every variable that occurs in f.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+1) \\ &= (2x+1)^2 - 5(2x+1) + 1 \\ &= 4x^2 + 4x + 1 - 10x - 5 + 1 \\ &= 4x^2 - 6x - 3\end{aligned}$$

To find $(g \circ f)(x)$ we will substitute f into every variable that occurs in g.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2 - 5x + 1) \\ &= 2(x^2 - 5x + 1) + 1 \\ &= 2x^2 - 10x + 2 + 1 \\ &= 2x^2 - 10x + 3\end{aligned}$$

$$\boxed{(f \circ g)(x) = 4x^2 - 6x - 3}$$

$$\boxed{(g \circ f)(x) = 2x^2 - 10x + 3}$$

3. **Given** $f(x) = 3x^2 + 2x - 5$ **and** $g(x) = 2x - 3$**, find** $(f \circ g)(x)$ **and** $(g \circ f)(x)$.

To find $(f \circ g)(x)$ we will substitute g in for every variable that occurs in f.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x-3) \\ &= 3(2x-3)^2 + 2(2x-3) - 5 \\ &= 3(4x^2 - 12x + 9) + 4x - 6 - 5 \\ &= 12x^2 - 36x + 27 + 4x - 6 - 5 \\ &= 12x^2 - 32x + 16\end{aligned}$$

To find $(g \circ f)(x)$ we will substitute f into every variable that occurs in g.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x^2 + 2x - 5) \\ &= 2(3x^2 + 2x - 5) - 3 \\ &= 6x^2 + 4x - 10 - 3 \\ &= 6x^2 + 4x - 13\end{aligned}$$

$$\boxed{(f \circ g)(x) = 12x^2 - 32x + 16}$$

$$\boxed{(g \circ f)(x) = 6x^2 + 4x - 13}$$

4. **Given** $f(x) = 2x^2 - 4x$ **and** $g(x) = x^2 + 1$**, find** $(f \circ g)(x)$ **and** $(g \circ f)(x)$.

To find $(f \circ g)(x)$ we will substitute g in for every variable that occurs in f.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2 + 1) \\ &= 2(x^2+1)^2 - 4(x^2+1) \\ &= 2(x^4 + 2x^2 + 1) - 4x^2 - 4 \\ &= 2x^4 + 4x^2 + 2 - 4x^2 - 4 \\ &= 2x^4 - 2\end{aligned}$$

To find $(g \circ f)(x)$ we will substitute f into every variable that occurs in g.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x^2 - 4x) \\ &= (2x^2 - 4x)^2 + 1 \\ &= 4x^4 - 16x^3 + 16x^2 + 1\end{aligned}$$

$$\boxed{(f \circ g)(x) = 2x^4 - 2}$$

$$\boxed{(g \circ f)(x) = 4x^4 - 16x^3 + 16x^2 + 1}$$

5. **Given** $f(x) = \frac{x}{x+1}$ **and** $g(x) = 9x - 3$**, find** $(f \circ g)(x)$ **and** $(g \circ f)(x)$.

To find $(f \circ g)(x)$ we will substitute $g(x) = 9x - 3$ in for every variable that occurs in f.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(9x-3) \\ &= \frac{9x-3}{9x-3+1} \\ &= \frac{9x-3}{9x-2}\end{aligned}$$

To find $(g \circ f)(x)$ we will substitute $f(x) = \frac{x}{x+1}$ into every variable that occurs in g.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g\left(\frac{x}{x+1}\right) \\ &= 9\left(\frac{x}{x+1}\right) - 3 \\ &= \frac{9x}{x+1} - 3 \\ &= \frac{9x}{x+1} - \frac{3(x+1)}{x+1} \\ &= \frac{9x-3(x+1)}{x+1} \\ &= \frac{9x-3x-3}{x+1} \\ &= \frac{6x-3}{x+1}\end{aligned}$$

$$\boxed{(f \circ g)(x) = \frac{9x-3}{9x-2}}$$

$$\boxed{(g \circ f)(x) = \frac{6x-3}{x+1}}$$

6. **Given** $f(x) = 6x - 7$ **and** $g(x) = x^2 + 3x + 5$**, find**

(a) $(g \circ f)(-1)$

We know that $(g \circ f)(-1) = g(f(-1))$. Therefore, we will first find $f(-1)$.

$$\begin{aligned} f(-1) &= 6(-1) - 7 \\ &= -6 - 7 \\ &= -13 \end{aligned}$$

Now, we will substitute -13 into every variable that occurs in g.

$$\begin{aligned} g(f(-1)) &= g(-13) \\ &= (-13)^2 + 3(-13) + 5 \\ &= 169 - 39 + 5 \\ &= 135 \end{aligned}$$

$$\boxed{(g \circ f)(-1) = 135}$$

(b) $(f \circ f)(2)$

We know that $(f \circ f)(2) = f(f(2))$. Hence, we first will find $f(2)$.

$$\begin{aligned} f(2) &= 6(2) - 7 \\ &= 12 - 7 \\ &= 5 \end{aligned}$$

Now, we will substitute 5 into every variable that occurs in f.

$$\begin{aligned} (f \circ f)(2) &= f(f(2)) \\ &= f(5) \\ &= 6(5) - 7 \\ &= 30 - 7 \\ &= 23. \end{aligned}$$

$$\boxed{(f \circ f)(2) = 23}$$

(c) $(g \circ g)(0)$

We know that $(g \circ g)(0) = g(g(0))$. So, we first need to find $g(0)$.

$$\begin{aligned} g(0) &= 0^2 + 3(0) + 5 \\ &= 5 \end{aligned}$$

Now, we will substitute 5 into every variable that occurs in g.

$$\begin{aligned} (g \circ g)(0) &= g(g(0)) \\ &= g(5) \\ &= 5^2 + 3(5) + 5 \\ &= 25 + 15 + 5 \\ &= 45. \end{aligned}$$

$$\boxed{(g \circ g)(0) = 45}$$

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7. **Express the function in the form $f \circ g$.**

(a) $F(x) = \sqrt{x - 7}$

Because we are looking for the form $f \circ g$, we know that f is the outside function and g is the inside function. Therefore, one answer is

$$\boxed{f(x) = \sqrt{x}, \qquad g(x) = x - 7}$$

(b) $F(x) = \dfrac{3}{x - 5}$

Once again, f is the outside function and g is the inside function. Therefore, one answer is

$$\boxed{f(x) = \frac{3}{x}, \qquad g(x) = x - 5}$$