Definition:

- **Composition function**: Given two functions \( f \) and \( g \), the composition function \( f \circ g \) is defined by

\[
(f \circ g)(x) = f(g(x)).
\]

In other words, given a number \( x \), we first apply \( g \) to it and then we apply \( f \) to the result. Here, \( f \) is the outside function and \( g \) is the inside function.

Important Properties:

- Let \( c \) be any constant. There are two ways to find \((f \circ g)(c)\). You could first evaluate \( g(c) \) and then evaluate \( f \) at the result. Or you could first find \((f \circ g)(x)\) and then evaluate the resulting function at \( c \).
- To find \((f \circ g)(x)\), remember to substitute the value \( g(x) \) into every variable that occurs in \( f \).
- The order of the functions is important. In general,

\[
(f \circ g)(x) \neq (g \circ f)(x).
\]

- Other composition functions are defined similarly. Namely,

\[
\begin{align*}
(g \circ f)(x) &= g(f(x)) \\
(f \circ f)(x) &= f(f(x)) \\
(g \circ g)(x) &= g(g(x))
\end{align*}
\]

Common Mistakes to Avoid:

- Composition of functions is different than the multiplication of functions. Therefore,

\[
(f \circ g)(x) \neq f(x) \cdot g(x).
\]
PROBLEMS

1. If \( f(x) = 3x - 5 \) and \( g(x) = x + 2 \) find \((f \circ g)(x)\) and \((g \circ f)(x)\).

To find \((f \circ g)(x)\) we will substitute \( g \) in for every variable that occurs in \( f \).

\[
(f \circ g)(x) = f(g(x)) \\
= f(x + 2) \\
= 3(x + 2) - 5 \\
= 3x + 6 - 5 \\
= 3x + 1
\]

To find \((g \circ f)(x)\) we will substitute \( f \) into every variable that occurs in \( g \).

\[
(g \circ f)(x) = g(f(x)) \\
= g(3x - 5) \\
= (3x - 5) + 2 \\
= 3x - 3
\]

\[
(f \circ g)(x) = 3x + 1 \\
(g \circ f)(x) = 3x - 3
\]

2. Given \( f(x) = x^2 - 5x + 1 \) and \( g(x) = 2x + 1 \), find \((f \circ g)(x)\) and \((g \circ f)(x)\).

To find \((f \circ g)(x)\) we will substitute \( g \) in for every variable that occurs in \( f \).

\[
(f \circ g)(x) = f(g(x)) \\
= f(2x + 1) \\
= (2x + 1)^2 - 5(2x + 1) + 1 \\
= 4x^2 + 4x + 1 - 10x - 5 + 1 \\
= 4x^2 - 6x - 3
\]

To find \((g \circ f)(x)\) we will substitute \( f \) into every variable that occurs in \( g \).

\[
(g \circ f)(x) = g(f(x)) \\
= g(x^2 - 5x + 1) \\
= 2(x^2 - 5x + 1) + 1 \\
= 2x^2 - 10x + 2 + 1 \\
= 2x^2 - 10x + 3
\]

\[
(f \circ g)(x) = 4x^2 - 6x - 3 \\
(g \circ f)(x) = 2x^2 - 10x + 3
\]
3. Given \( f(x) = 3x^2 + 2x - 5 \) and \( g(x) = 2x - 3 \), find \((f \circ g)(x)\) and \((g \circ f)(x)\).

To find \((f \circ g)(x)\) we will substitute \(g\) in for every variable that occurs in \(f\).

\[
(f \circ g)(x) = f(g(x))
= f(2x - 3)
= 3(2x - 3)^2 + 2(2x - 3) - 5
= 3(4x^2 - 12x + 9) + 4x - 6 - 5
= 12x^2 - 36x + 27 + 4x - 6 - 5
= 12x^2 - 32x + 16
\]

To find \((g \circ f)(x)\) we will substitute \(f\) into every variable that occurs in \(g\).

\[
(g \circ f)(x) = g(f(x))
= g(3x^2 + 2x - 5)
= 2(3x^2 + 2x - 5) - 3
= 6x^2 + 4x - 10 - 3
= 6x^2 + 4x - 13
\]

\[
(f \circ g)(x) = 12x^2 - 32x + 16
\]

\[
(g \circ f)(x) = 6x^2 + 4x - 13
\]

4. Given \( f(x) = 2x^2 - 4x \) and \( g(x) = x^2 + 1 \), find \((f \circ g)(x)\) and \((g \circ f)(x)\).

To find \((f \circ g)(x)\) we will substitute \(g\) in for every variable that occurs in \(f\).

\[
(f \circ g)(x) = f(g(x))
= f(x^2 + 1)
= 2(x^2 + 1)^2 - 4(x^2 + 1)
= 2x^4 + 2x^2 + 1 - 4x^2 - 4
= 2x^4 + 4x^2 + 2 - 4x^2 - 4
= 2x^4 - 2
\]

To find \((g \circ f)(x)\) we will substitute \(f\) into every variable that occurs in \(g\).

\[
(g \circ f)(x) = g(f(x))
= g(2x^2 - 4x)
= (2x^2 - 4x)^2 + 1
= 4x^4 - 16x^3 + 16x^2 + 1
\]

\[
(f \circ g)(x) = 2x^4 - 2
\]

\[
(g \circ f)(x) = 4x^4 - 16x^3 + 16x^2 + 1
\]
5. **Given** \( f(x) = \frac{x}{x+1} \) and \( g(x) = 9x - 3 \), find \((f \circ g)(x)\) and \((g \circ f)(x)\).

To find \((f \circ g)(x)\) we will substitute \(g(x) = 9x - 3\) in for every variable that occurs in \(f\).

\[
(f \circ g)(x) = f(g(x)) = f(9x - 3) = \frac{9x - 3}{9x - 3 + 1} = \frac{9x - 3}{9x - 2}
\]

To find \((g \circ f)(x)\) we will substitute \(f(x) = \frac{x}{x+1}\) into every variable that occurs in \(g\).

\[
(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{x+1}\right) = 9\left(\frac{x}{x+1}\right) - 3 = \frac{9x}{x+1} - 3 = \frac{9x - 3(x + 1)}{x+1} = \frac{9x - 3x - 3}{x + 1} = \frac{6x - 3}{x + 1}
\]

\[
(f \circ g)(x) = \frac{9x - 3}{9x - 2} \\
(g \circ f)(x) = \frac{6x - 3}{x + 1}
\]
6. **Given** \( f(x) = 6x - 7 \) and \( g(x) = x^2 + 3x + 5 \), find

(a) \((g \circ f)(-1)\)

We know that \((g \circ f)(-1) = g(f(-1))\). Therefore, we will first find \(f(-1)\).

\[
f(-1) = 6(-1) - 7
= -6 - 7
= -13.
\]

Now, we will substitute \(-13\) into every variable that occurs in \(g\).

\[
g(f(-1)) = g(-13)
= (-13)^2 + 3(-13) + 5
= 169 - 39 + 5
= 135
\]

\[
(g \circ f)(-1) = 135
\]

(b) \((f \circ f)(2)\)

We know that \((f \circ f)(2) = f(f(2))\). Hence, we first will find \(f(2)\).

\[
f(2) = 6(2) - 7
= 12 - 7
= 5.
\]

Now, we will substitute \(5\) into every variable that occurs in \(f\).

\[
(f \circ f)(2) = f(f(2))
= f(5)
= 6(5) - 7
= 30 - 7
= 23.
\]

\[
(f \circ f)(2) = 23
\]
(c) \((g \circ g)(0)\)

We know that \((g \circ g)(0) = g(g(0))\). So, we first need to find \(g(0)\).

\[
g(0) = 0^2 + 3(0) + 5
= 5
\]

Now, we will substitute 5 into every variable that occurs in \(g\).

\[
(g \circ g)(0) = g(g(0))
= g(5)
= 5^2 + 3(5) + 5
= 25 + 15 + 5
= 45.
\]

\( (g \circ g)(0) = 45 \)

7. Express the function in the form \(f \circ g\).

(a) \(F(x) = \sqrt{x - 7}\)

Because we are looking for the form \(f \circ g\), we know that \(f\) is the outside function and \(g\) is the inside function. Therefore, one answer is

\[
f(x) = \sqrt{x}, \quad g(x) = x - 7
\]

(b) \(F(x) = \frac{3}{x - 5}\)

Once again, \(f\) is the outside function and \(g\) is the inside function. Therefore, one answer is

\[
f(x) = \frac{3}{x}, \quad g(x) = x - 5
\]