Definition:

- **Difference quotient**: is an expression of the form

$$\frac{f(a+h)-f(a)}{h}.$$

They represent the average change in the value of f between $x=a$ and $x=a+h$. They are used in calculus.

Important Properties:

- In the numerator of a difference quotient, any term that does not contain a h must subtract off.
- When evaluating functions remember that whatever is inside the parenthesis, regardless of what it looks like, is substituted into every variable x. For example, if $f(x)=x^2-3x+1$ then

$$\begin{aligned} f(a+h) &= (a+h)^2-3(a+h)+1 \\ &= a^2+2ah+h^2-3a-3h+1 \end{aligned}$$

- When evaluating a difference quotient, the h in the denominator will always divide out.

Common Mistakes to Avoid:

- Note that $f(a+h)\neq f(a)+h$.
- Remember that $(a+h)^2\neq a^2+h^2$. Instead,

$$(a+h)^2=a^2+2ah+h^2$$

by using foil.

- Do NOT distribute inside a quantity raised to a power. Remember to raise a quantity to its power *before* you distribute. For example, $3(a+h)^2\neq(3a+3h)^2$. Instead,

$$3(a+h)^2=3(a^2+2ah+h^2)=3a^2+6ah+3h^2.$$

PROBLEMS

Find $\frac{f(a+h)-f(a)}{h}$ for each of the given functions.

1. $f(x) = x + 2$

Here, we have that

$$f(a) = a + 2$$
$$f(a+h) = a + h + 2.$$

Substituting these into the difference quotient, we get

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{a+h+2-(a+2)}{h}\\ &= \frac{a+h+2-a-2}{h}\\ &= \frac{h}{h}\\ &= 1\end{aligned}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = 1}$$

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2. $f(x) = 3x - 5$

Here, we have that

$$\begin{aligned}f(a) &= 3a - 5\\ f(a+h) &= 3(a+h) - 5\\ &= 3a + 3h - 5.\end{aligned}$$

Substituting these into the difference quotient, we get

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{3a+3h-5-(3a-5)}{h}\\ &= \frac{3a+3h-5-3a+5}{h}\\ &= \frac{3h}{h}\\ &= 3\end{aligned}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = 3}$$

3. $f(x) = x^2 - 4$

   Here, we have that

$$\begin{aligned} f(a) &= a^2 - 4 \\ f(a+h) &= (a+h)^2 - 4 \\ &= a^2 + 2ah + h^2 - 4 \end{aligned}$$

   Substituting these into the difference quotient, we get

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{a^2 + 2ah + h^2 - 4 - (a^2 - 4)}{h} \\ &= \frac{a^2 + 2ah + h^2 - 4 - a^2 + 4}{h} \\ &= \frac{2ah + h^2}{h} \\ &= \frac{h(2a+h)}{h} \\ &= 2a + h \end{aligned}$$

$$\boxed{\frac{f(a+h) - f(a)}{h} = 2a + h}$$

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4. $f(x) = x^2 - 5x + 7$

   We know that

$$\begin{aligned} f(a) &= a^2 - 5a + 7 \\ f(a+h) &= (a+h)^2 - 5(a+h) + 7 \\ &= a^2 + 2ah + h^2 - 5a - 5h + 7 \end{aligned}$$

   Substituting these into the difference quotient, we get

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{a^2+2ah+h^2-5a-5h+7-(a^2-5a+7)}{h}\\ &= \frac{a^2+2ah+h^2-5a-5h+7-a^2+5a-7}{h}\\ &= \frac{2ah+h^2-5h}{h}\\ &= \frac{h(2a+h-5)}{h}\\ &= 2a+h-5\end{aligned}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = 2a+h-5}$$

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5. $f(x) = 4x^2+3x-2$

   We know that

$$\begin{aligned}f(a) &= 4a^2+3a-2\\ f(a+h) &= 4(a+h)^2+3(a+h)-2\\ &= 4(a^2+2ah+h^2)+3a+3h-2\\ &= 4a^2+8ah+4h^2+3a+3h-2\end{aligned}$$

   Substituting these into the difference quotient, we get

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{4a^2+8ah+4h^2+3a+3h-2-(4a^2+3a-2)}{h}\\ &= \frac{4a^2+8ah+4h^2+3a+3h-2-4a^2-3a+2}{h}\\ &= \frac{8ah+4h^2+3h}{h}\\ &= \frac{h(8a+4h+3)}{h}\\ &= 8a+4h+3\end{aligned}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = 8a+4h+3}$$

6. $f(x) = \dfrac{1}{x}$

Here, we have that

$$f(a) = \frac{1}{a}$$
$$f(a+h) = \frac{1}{a+h}$$

Substituting these into the difference quotient, we get

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} \\
&= \frac{\dfrac{a}{a(a+h)} - \dfrac{(a+h)}{a(a+h)}}{h} \\
&= \frac{\dfrac{a-(a+h)}{a(a+h)}}{h} \\
&= \frac{\dfrac{a-a-h}{a(a+h)}}{h} \\
&= \frac{\dfrac{-h}{a(a+h)}}{h} \\
&= \frac{-h}{ah(a+h)} \\
&= \frac{-1}{a(a+h)}
\end{aligned}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = \frac{-1}{a(a+h)}}$$

7. $f(x) = \dfrac{3}{x+2}$

   Here, we have that

$$f(a) = \frac{3}{a+2}$$
$$f(a+h) = \frac{3}{a+h+2}$$

   Substituting these into the difference quotient, we get

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\dfrac{3}{a+h+2} - \dfrac{3}{a+2}}{h} \\
&= \frac{\dfrac{3(a+2)}{(a+2)(a+h+2)} - \dfrac{3(a+h+2)}{(a+2)(a+h+2)}}{h} \\
&= \frac{\dfrac{3(a+2)-3(a+h+2)}{(a+2)(a+h+2)}}{h} \\
&= \frac{\dfrac{3a+6-3a-3h-6}{(a+2)(a+h+2)}}{h} \\
&= \frac{\dfrac{-3h}{(a+2)(a+h+2)}}{h} \\
&= \frac{-3h}{h(a+2)(a+h+2)} \\
&= \frac{-3}{(a+2)(a+h+2)}
\end{aligned}$$

$$\boxed{\frac{f(a+h)-f(a)}{h} = \frac{-3}{(a+2)(a+h+2)}}$$