EXPONENTIAL FUNCTIONS AND THEIR APPLICATIONS

Definitions:

- **Exponential function**: For $a > 0$, the exponential function with base a is defined by

$$\boxed{f(x) = a^x}$$

- **Horizontal asymptote**: The line $y = c$ is a horizontal asymptote of the function f if

$$f(x) \to c \qquad \text{as} \qquad x \to \infty \quad \text{or} \quad x \to -\infty.$$

Properties of the graph of $f(x) = a^x, \ a > 0$

- Domain is all real numbers.
- Range is $(0, \infty)$.
- Always crosses through the point $(0, 1)$.
- $y = 0$ is a horizontal asymptote.
- If $a > 1$, then the function is increasing; if $0 < a < 1$, then the function is decreasing.

Important Formulas: Exponential functions are used in a variety of important formulas.

- **Compound Interest**: is calculated by the formula

$$\boxed{A(t) = P\left(1 + \frac{r}{n}\right)^{nt}}$$

where

$$\begin{aligned} A(t) &= \text{ amount after } t \text{ years} \\ P &= \text{ principal} \\ r &= \text{ interest rate} \\ n &= \text{ number of times the interest is compounded per year} \\ t &= \text{ number of years} \end{aligned}$$

- **Compounded Continuously Interest**: is calculated by the formula

$$\boxed{A(t) = Pe^{rt}}$$

where

$$\begin{aligned} A(t) &= \text{ amount after } t \text{ years} \\ P &= \text{ principal} \\ r &= \text{ interest rate} \\ t &= \text{ number of years} \end{aligned}$$

- **Exponential Growth**: of a population increases according to the formula

$$\boxed{P(t) = P_0 e^{rt}}$$

  where

$$\begin{aligned} P(t) &= \text{ population after time } t \\ P_0 &= \text{ initial population} \\ r &= \text{ growth rate} \\ t &= \text{ time} \end{aligned}$$

Important Properties:

- Every exponential function is a one-to-one function and hence has an inverse.

Common Mistakes to Avoid:

- Do NOT use the compounded continuously formula unless it says *compounded continuously* in the problem.
- In the exponential growth and compounded continuously formulas the rt is the exponent on e. Do NOT multiply e by rt.
- Remember to convert all interest or growth rates to a decimal before substituting into a formula.

PROBLEMS

1. **If $\$5,000$ is invested at a rate of 8%, compounded weekly, find the value of the investment after 7 years.**

   Here $P = 5000,\ t = 7,\ r = .08$ and $n = 52$ since there are 52 weeks in a year. Substituting these values into our compound interest formula, we get

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A(7) = 5000\left(1 + \frac{.08}{52}\right)^{52\cdot 7}$$
$$= 5000\,(1.001538462)^{364}$$
$$= 8749.596496$$

$\boxed{\$8749.60}$

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2. **If $\$4000$ is borrowed at a rate of 16% interest per year, compounded quarterly, find the amount due at the end of 4 years? 8 years?**

   Here $P = 4000,\ r = .16$ and $n = 4$ since there are 4 quarters in a year. To find the amount due at the end of 4 years we let $t = 4$. Substituting into the compound interest formula, we get

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A(4) = 4000\left(1 + \frac{.16}{4}\right)^{4\cdot 4}$$
$$= 4000\,(1.04)^{16}$$
$$= 7491.924983$$

$\boxed{\$7491.92 \text{ due at the end of 4 years}}$

   To find the amount due at the end of 8 years, we change $t = 8$.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A(8) = 4000\left(1 + \frac{.16}{4}\right)^{4\cdot 8}$$
$$= 4000\,(1.04)^{32}$$
$$= 14032.23499$$

$\boxed{\$14032.23 \text{ due at the end of 8 years}}$

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3. **If $\$3000$ is borrowed at a rate of 12% interest per year, find the amount due at the end of 5 years if the interest is compounded annually? monthly? daily?**

   For this problem, we have $P = 3000,\ r = .12$ and $t = 5$.

   If the money is compounded annually, then $n = 1$. Substituting into the compound interest formula, we get

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A(5) = 3000\left(1 + \frac{.12}{1}\right)^{1\cdot 5}$$
$$= 3000\,(1.12)^{5}$$
$$= 5287.02505$$

$\boxed{\$5287.03 \text{ due if compounded annually}}$

When the money is compounded monthly, $n = 12$ since there are 12 months in one year. Therefore,

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A(5) = 3000\left(1 + \frac{.12}{12}\right)^{12\cdot 5}$$
$$= 3000\,(1.01)^{60}$$
$$= 5450.090096$$

$\boxed{\$5450.09 \text{ due if compounded monthly}}$

Finally, if the money is compounded daily, then $n = 365$ since there are 365 days in one year. Hence,

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A(5) = 3000\left(1 + \frac{.12}{365}\right)^{365\cdot 5}$$
$$= 3000\,(1.000328767)^{1825}$$
$$= 5465.817399$$

$\boxed{\$5465.82 \text{ due if compounded daily}}$

4. **If \$3000 is borrowed at a rate of 12% interest per year, find the amount due at the end of 5 years if the interest is compounded continuously.**

For this problem, we use the compounded continuously formula with $P = 3000$, $r = .12$, and $t = 5$. Substituting everything in, we get

$$A(t) = Pe^{rt}$$
$$A(5) = 3000e^{.12\cdot 5}$$
$$= 3000(1.8221188)$$
$$= 5466.356401$$

$\boxed{\$5466.36 \text{ due if compounded continuously}}$

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5. **Find the amount that must be invested at $5\frac{1}{2}\%$ today in order to have \$100,000 in 20 years if the investment is compounded continuously.**

Here, we have $A(20) = 100,000$, $r = .055$, and $t = 20$. Substituting into the compounded continuously interest formula, we get

$$A(t) = Pe^{rt}$$
$$100,000 = Pe^{.055\cdot 20}$$
$$100,000 = Pe^{1.1}$$
$$\frac{100,000}{e^{1.1}} = P$$
$$33287.10837 = P$$

$\boxed{\$33,287.11 \text{ must be invested now}}$

6. **The population of a certain city has a relative growth rate of 9% per year. The population in 1978 was 24,000. Find the projected population of the city for the year 2010.**

Since 1978 is our starting date, 2010 refers to $t = 22$. Also, we know that $P_0 = 24,000$ and $r = .09$. Substituting into our exponential growth formula, we get

$$\begin{aligned} P(t) &= P_0 e^{rt} \\ P(22) &= 24,000e^{.09 \cdot 22} \\ &= 24,000e^{1.98} \\ &= 173825.8316 \end{aligned}$$

$\boxed{173,825 \text{ people in 2010}}$

7. **The relative growth rate for a certain bacteria population is 75% per hour. A small culture is formed and 4 hours later a count shows approximately 32,500 bacteria in a culture. Find the initial number of bacteria in the culture and estimate the number of bacteria 6 hours from the time the culture was started.**

To solve for the initial number of bacteria in the culture, we will use the exponential growth formula with $r = .75$, $t = 4$, and $P(4) = 32,500$. Therefore,

$$\begin{aligned} P(t) &= P_0 e^{rt} \\ 32,500 &= P_0 e^{.75 \cdot 4} \\ 32,500 &= P_0 e^{3} \\ \frac{32,500}{e^3} &= P_0 \\ 1618.079722 &= P_0 \end{aligned}$$

$\boxed{1618 \text{ bacteria initially}}$

Since we have the initial population of 1618, substituting this into our exponential growth formula with $t = 6$, we get

$$\begin{aligned} P(t) &= P_0 e^{rt} \\ P(6) &= 1618e^{.75 \cdot 6} \\ &= 1618e^{4.5} \\ &= 145647.7184 \end{aligned}$$

$\boxed{145,647 \text{ bacteria after 6 hours}}$