Definitions:

- **Rectangular Coordinate System**: consists of a vertical line called the $y$–axis and a horizontal line called the $x$–axis. The $x$–axis and $y$–axis divide the coordinate plane into four quadrants and intersect at a point called the **origin**. Each point in the plane corresponds to a unique ordered pair $(x, y)$.

- **Midpoint**: of a line segment $AB$ is the point that is equidistant from the endpoints $A$ and $B$.

Important Properties:

- **Distance formula**: Suppose that $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are two points in the coordinate plane. The distance between $A$ and $B$, denoted $d(A, B)$, is given by

$$d(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$  

It does not matter in what order you subtract the $x$–coordinates or the $y$–coordinates.

- **Midpoint formula**: Suppose that $A = (x_1, y_1)$ and $B = (x_2, y_2)$ are the endpoints of the line segment $AB$. Then the midpoint $M$ of $AB$ is given by

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

- **Pythagorean Theorem**: In a right triangle, if the side opposite the right angle has length $c$ and the other two sides have lengths $a$ and $b$, then

$$a^2 + b^2 = c^2.$$  

The side opposite the right angle is called the **hypotenuse** and the other two sides are called **legs**.

- The converse of the Pythagorean Theorem is also true. Namely, if a triangle has sides of lengths $a, b$ and $c$ which satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Common Mistakes to Avoid:

- The midpoint is found by **averaging** the $x$–coordinates and **averaging** the $y$–coordinates. Do NOT subtract them.

- The square root of a sum is NOT the sum of the square roots. In other words,

$$\sqrt{x^2 + y^2} \neq \sqrt{x^2} + \sqrt{y^2}.$$  

To illustrate this with an example, notice that

$$\sqrt{9 + 16} = \sqrt{25} = 5 \quad \text{whereas} \quad \sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

- When using the Pythagorean Theorem, make sure that the hypotenuse is on a side by itself; namely,

$$\text{leg}^2 + \text{leg}^2 = \text{hypotenuse}^2.$$
PROBLEMS

1. Find the distance between the given points.

(a) $A = (2,0)$ and $B = (0,9)$

$$d(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(2 - 0)^2 + (0 - 9)^2}$$

$$= \sqrt{2^2 + (-9)^2}$$

$$= \sqrt{4 + 81}$$

$$= \sqrt{85}$$

$$d(A, B) = \sqrt{85}$$

(b) $A = (-2,5)$ and $B = (12,3)$

$$d(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(-2 - 12)^2 + (5 - 3)^2}$$

$$= \sqrt{(-14)^2 + (2)^2}$$

$$= \sqrt{196 + 4}$$

$$= \sqrt{200}$$

$$= \sqrt{100 \cdot 2}$$

$$= 10\sqrt{2}$$

$$d(A, B) = 10\sqrt{2}$$

(c) $A = (-2,3)$ and $B = (9,-3)$

$$d(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(-2 - 9)^2 + (3 - (-3))^2}$$

$$= \sqrt{(-11)^2 + (6)^2}$$

$$= \sqrt{121 + 36}$$

$$= \sqrt{157}$$

$$d(A, B) = \sqrt{157}$$

(d) $A = (-1,-5)$ and $B = (-4,7)$

$$d(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(-1 - (-4))^2 + (-5 - 7)^2}$$

$$= \sqrt{(3)^2 + (-12)^2}$$

$$= \sqrt{9 + 144}$$

$$= \sqrt{153}$$

$$= \sqrt{9 \cdot 17}$$

$$= 3\sqrt{17}$$

$$d(A, B) = 3\sqrt{17}$$
2. Find the midpoint \( M \) of the line segment \( AB \) where

(a) \( A = (8, -4) \) and \( B = (-2, 2) \)

\[
M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\
= \left( \frac{8 - 2}{2}, \frac{-4 + 2}{2} \right) \\
= \left( \frac{6}{2}, \frac{-2}{2} \right) \\
= (3, -1)
\]

\[ M = (3, -1) \]

(b) \( A = (5, -6) \) and \( B = (-2, 11) \)

\[
M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\
= \left( \frac{5 - 2}{2}, \frac{-6 + 11}{2} \right) \\
= \left( \frac{3}{2}, \frac{5}{2} \right) \\
M = \left( \frac{3}{2}, \frac{5}{2} \right)
\]

3. If \( M = (3, -2) \) is the midpoint of the line segment \( AB \) and if \( A = (-9, 2) \) find the coordinates of \( B \).

Let \( B = (x, y) \). Then using the midpoint formula we get

\[
M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\
(3, -2) = \left( \frac{x + (-9)}{2}, \frac{y + 2}{2} \right)
\]

Therefore, equating coordinates, we find that

\[
\frac{x + (-9)}{2} = 3 \\
x + (-9) = 6 \\
x = 15
\]

\[
\frac{y + 2}{2} = -2 \\
y + 2 = -4 \\
y = -6
\]

\[ B = (15, -6) \]
4. Find the point on the line segment \( AB \) that is one-fourth of the distance from the point \( A = (3, -4) \) to the point \( B = (-5, 12) \).

First, we will find the midpoint of the line segment.

\[
M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{3 - 5}{2}, \frac{-4 + 12}{2} \right) = \left( \frac{-2}{2}, \frac{8}{2} \right) = (-1, 4)
\]

Now, \( M \) is one-half the distance from \( A \) to \( B \). Therefore, the point we need, let's call it \( C \), is the midpoint between \( A \) and \( M \).

\[
C = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{3 - 1}{2}, \frac{-4 + 4}{2} \right) = \left( \frac{2}{2}, \frac{0}{2} \right) = (1, 0)
\]

\[ C = (1, 0) \]

5. Determine the point \( C \) on the \( x \)-axis that is equidistant from \( A = (-2, 3) \) and \( B = (1, -5) \).

NOTE: Although we are looking for the point equidistant, we are NOT looking for the midpoint. Since we are told the point lies on the \( x \)-axis, we are looking for a point of the form \( C = (x, 0) \). Since they are equidistant, they must have the same distances. Therefore, we will equate the distance formulas.

\[
d(A, C) = d(B, C) \\
\sqrt{(-2 - x)^2 + (3 - 0)^2} = \sqrt{(1 - x)^2 + (-5 - 0)^2} \\
(-2 - x)^2 + (3)^2 = (1 - x)^2 + (-5)^2 \\
4 + 4x + x^2 + 9 = 1 - 2x + x^2 + 25 \\
x^2 + 4x + 13 = x^2 - 2x + 26 \\
4x + 13 = -2x + 26 \\
6x + 13 = 26 \\
x = \frac{13}{6}
\]

\[ C = \left( \frac{13}{6}, 0 \right) \]
6. Determine if \( A = (2, -3) \), \( B = (1, 8) \) and \( C = (-4, 2) \) are the vertices of a right triangle.

To solve this, we will first find the distances between the three points. Then we will check to see if they satisfy the Pythagorean Theorem.

\[
d(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
= \sqrt{(2 - 1)^2 + (-3 - 8)^2} \\
= \sqrt{(1)^2 + (-11)^2} \\
= \sqrt{1 + 121} \\
= \sqrt{122}
\]

\[
d(A, C) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
= \sqrt{(2 - (-4))^2 + (-3 - 2)^2} \\
= \sqrt{(6)^2 + (-5)^2} \\
= \sqrt{36 + 25} \\
= \sqrt{61}
\]

\[
d(B, C) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
= \sqrt{(1 - (-4))^2 + (8 - 2)^2} \\
= \sqrt{(5)^2 + (6)^2} \\
= \sqrt{25 + 36} \\
= \sqrt{61}
\]

Now, substituting into \( a^2 + b^2 = c^2 \), we find

\[
a^2 + b^2 = c^2 \\
(\sqrt{61})^2 + (\sqrt{61})^2 = (\sqrt{122})^2 \\
61 + 61 = 122 \\
122 = 122*
\]

Therefore, by the converse of the Pythagorean Theorem,

\[
\text{Triangle } ABC \text{ is a right triangle.}
\]