Definitions:

• \textbf{\textit{n-th roots of } }a: The \textit{n}-th root of \textit{a}, denoted $\sqrt[n]{a}$, is a number whose \textit{n}-th power equals \textit{a}. In other words,

$$\sqrt[n]{a} = b \text{ means } b^n = a.$$ 

The number \textit{n} is called the \textbf{index}.

• \textbf{Rationalizing the denominator:} is the process of removing radicals from the denominator so that the expression will be in simplified form.

Important Properties:

• \textbf{Quotient rule for radicals:} If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and \textit{n} is a positive integer,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$ 

In other words, the radical of a quotient is the quotient of the radicals.

• Whenever a radical expression contains a sum or difference involving radicals in the denominator, we rationalize the denominator by multiplying both numerator and denominator by the conjugate of the denominator. The \textbf{conjugate} contains exactly the same numbers in exactly the same order with the operation sign changed. For example, the conjugate of $2 + \sqrt{3}$ is $2 - \sqrt{3}$ and the conjugate of $\sqrt{3} + 7$ is $\sqrt{3} - 7$.

Simplifying radicals: A radical is in simplest form when the following conditions are satisfied.

• The quantity under the radical has no factor raised to a power greater than or equal to the index.

• There is no fraction under the radical.

• There is no radical in the denominator.

• There is no common factor, other than 1, between the exponents on factors under the radical and the index.

Common Mistakes to Avoid:

• Be careful when rationalizing radical expressions that involve \textit{n}-th roots. For example,

$$\sqrt[3]{2} \cdot \sqrt[3]{2} \neq 2.$$ 

• Do not divide (or cancel) terms that are inside a radical with those that are outside a radical. For example,

$$\frac{\sqrt{6}}{15} \neq \frac{\sqrt{2}}{5}.$$
PROBLEMS

Rationalize the denominator in each radical expression.
Assume that all variables represent positive real numbers.

1. \( \frac{2}{\sqrt{3}} \)

We need to multiply both the numerator and denominator by \( \sqrt{3} \).

\[
\frac{2}{\sqrt{3}} = \frac{2 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \frac{2\sqrt{3}}{3}
\]

2. \( \frac{-6}{\sqrt{18}} \)

First, note that \( \sqrt{18} = \sqrt{9\sqrt{2}} = 3\sqrt{2} \). Therefore, we really only need to multiply the numerator and denominator by \( \sqrt{2} \) in order to rationalize.

\[
\frac{-6}{\sqrt{18}} = \frac{-6}{3\sqrt{2}} = \frac{-6 \cdot \sqrt{2}}{3\sqrt{2} \cdot \sqrt{2}} = \frac{-6\sqrt{2}}{3 \cdot 2} = \frac{-6\sqrt{2}}{6} = -\sqrt{2}
\]

3. \( \sqrt{\frac{5}{6}} \)

First, we will use the quotient rule for radicals and then multiply both numerator and denominator by \( \sqrt{6} \).

\[
\sqrt{\frac{5}{6}} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{5} \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} = \frac{\sqrt{30}}{\sqrt{36}} = \frac{\sqrt{30}}{6}
\]

4. \( \frac{9}{\sqrt{27}} \)

First note that \( \sqrt{27} = \sqrt{9\sqrt{3}} = 3\sqrt{3} \). Therefore, we only need to multiply the numerator and denominator by \( \sqrt{3} \) to rationalize.

\[
\frac{9}{\sqrt{27}} = \frac{9}{\sqrt{9}\sqrt{3}} = \frac{9}{3\sqrt{3}} = \frac{9 \cdot \sqrt{3}}{3\sqrt{3} \cdot \sqrt{3}} = \frac{9\sqrt{3}}{3 \cdot 3} = \frac{9\sqrt{3}}{9} = \frac{9\sqrt{3}}{9} = \sqrt{3}
\]

NOTE: You can also solve this problem by multiplying both the numerator and denominator by \( \sqrt{18} \) first and then simplify later.
5. \( \frac{5}{\sqrt{2}} \)

Notice that multiplying by \( \frac{\sqrt{2}}{\sqrt{2}} \) will NOT eliminate the radical from the denominator. Instead, we need to multiply by \( \frac{\sqrt{4}}{\sqrt{4}} \) on both top and bottom.

\[
\frac{5}{\sqrt{2}} = \frac{5 \cdot \sqrt{4}}{\sqrt{2} \cdot \sqrt{4}} = \frac{5 \sqrt{4}}{\sqrt{8}} = \frac{5 \sqrt{4}}{2}
\]

6. \( \frac{\sqrt{4}}{\sqrt{9}} \)

As in the previous example, we will NOT eliminate the radical in the denominator by multiplying by \( \frac{\sqrt{9}}{\sqrt{9}} \). Instead, we will multiply both numerator and denominator by \( \frac{\sqrt{3}}{\sqrt{3}} \).

\[
\frac{\sqrt{4}}{\sqrt{9}} = \frac{\sqrt{4} \cdot \sqrt{3}}{\sqrt{9} \cdot \sqrt{3}} = \frac{\sqrt{12}}{\sqrt{27}} = \frac{\sqrt{12}}{3}
\]

7. \( \frac{2}{\sqrt{5} - 3} \)

To rationalize we need to multiply both numerator and denominator by \( \sqrt{5} + 3 \) which is the conjugate of the denominator.

\[
\frac{2}{\sqrt{5} - 3} = \frac{2(\sqrt{5} + 3)}{(\sqrt{5} - 3)(\sqrt{5} + 3)} = \frac{2(\sqrt{5} + 3)}{\sqrt{25} + 3\sqrt{5} - 3\sqrt{5} - 9} = \frac{2(\sqrt{5} + 3)}{5 - 9} = \frac{2(\sqrt{5} + 3)}{-4} = -\frac{(\sqrt{5} + 3)}{2}
\]

8. \( \frac{1 - \sqrt{2}}{\sqrt{8} + \sqrt{6}} \)

To rationalize we need to multiply both numerator and denominator by \( \sqrt{8} - \sqrt{6} \) which is the conjugate of the denominator.

\[
\frac{1 - \sqrt{2}}{\sqrt{8} + \sqrt{6}} = \frac{(1 - \sqrt{2})(\sqrt{8} - \sqrt{6})}{(\sqrt{8} + \sqrt{6})(\sqrt{8} - \sqrt{6})} = \frac{(1 - \sqrt{2})(\sqrt{8} - \sqrt{6})}{\sqrt{64} + \sqrt{48} - \sqrt{48} + \sqrt{36}} = \frac{(1 - \sqrt{2})(\sqrt{8} - \sqrt{6})}{8 - 6} = \frac{(1 - \sqrt{2})(\sqrt{4\sqrt{2} - \sqrt{6}})}{2} = \frac{(1 - \sqrt{2})(2\sqrt{2} - \sqrt{6})}{2}
\]
9. \( \frac{4}{\sqrt{x} - 2\sqrt{y}} \)

We will rationalize by multiplying both numerator and denominator by \( \sqrt{x} + 2\sqrt{y} \).

\[
\frac{4}{\sqrt{x} - 2\sqrt{y}} = \frac{4(\sqrt{x} + 2\sqrt{y})}{(\sqrt{x} - 2\sqrt{y})(\sqrt{x} + 2\sqrt{y})} = \frac{4\sqrt{x} + 8\sqrt{y}}{x - 4y}
\]

10. \( \frac{15}{\sqrt{7} + \sqrt{2}} \)

We will rationalize by multiplying the numerator and denominator by \( \sqrt{7} - \sqrt{2} \).

\[
\frac{15}{\sqrt{7} + \sqrt{2}} = \frac{15(\sqrt{7} - \sqrt{2})}{(\sqrt{7} + \sqrt{2})(\sqrt{7} - \sqrt{2})} = \frac{15(\sqrt{7} - \sqrt{2})}{7 - 2} = \frac{15(\sqrt{7} - \sqrt{2})}{5} = 3(\sqrt{7} - \sqrt{2})
\]