Definitions:

- ***n*-th roots of *a***: The n-th root of a, denoted $\sqrt[n]{a}$, is a number whose n-th power equals a. In other words,

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a.$$

  The number n is called the **index**.

- **Rationalizing the denominator**: is the process of removing radicals from the denominator so that the expression will be in simplified form.

Important Properties:

- **Quotient rule for radicals**: If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a positive integer,

$$\boxed{\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.}$$

  In other words, the radical of a quotient is the quotient of the radicals.

- Whenever a radical expression contains a sum or difference involving radicals in the denominator, we rationalize the denominator by multiplying both numerator and denominator by the conjugate of the denominator. The **conjugate** contains exactly the same numbers in exactly the same order with the operation sign changed. For example, the conjugate of $2 + \sqrt{3}$ is $2 - \sqrt{3}$ and the conjugate of $\sqrt{3} + 7$ is $\sqrt{3} - 7$.

Simplifying radicals: A radical is in simplest form when the following conditions are satisfied.

- The quantity under the radical has no factor raised to a power greater than or equal to the index.
- There is no fraction under the radical.
- There is no radical in the denominator.
- There is no common factor, other than 1, between the exponents on factors under the radical and the index.

Common Mistakes to Avoid:

- Be careful when rationalizing radical expressions that involve n-th roots. For example,

$$\sqrt[3]{2} \cdot \sqrt[3]{2} \neq 2.$$

- Do not divide (or cancel) terms that are inside a radical with those that are outside a radical. For example,

$$\frac{\sqrt{6}}{15} \neq \frac{\sqrt{2}}{5}.$$

PROBLEMS

Rationalize the denominator in each radical expression.
Assume that all variables represent positive real numbers.

1. $\dfrac{2}{\sqrt{3}}$

We need to multiply both the numerator and denominator by $\sqrt{3}$.

$$\frac{2}{\sqrt{3}} = \frac{2\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \boxed{\frac{2\sqrt{3}}{3}}$$

---

2. $\dfrac{-6}{\sqrt{18}}$

First, note that $\sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$. Therefore, we really only need to multiply the numerator and denominator by $\sqrt{2}$ in order to rationalize.

$$\frac{-6}{\sqrt{18}} = \frac{-6}{3\sqrt{2}} = \frac{-6\cdot\sqrt{2}}{3\sqrt{2}\cdot\sqrt{2}} = \frac{-6\sqrt{2}}{3\sqrt{4}} = \frac{-6\sqrt{2}}{3\cdot 2} = \frac{-6\sqrt{2}}{6} = \boxed{-\sqrt{2}}$$

NOTE: You can also solve this problem by multiplying both the numerator and denominator by $\sqrt{18}$ first and then simplify later.

3. $\sqrt{\dfrac{5}{6}}$

First, we will use the quotient rule for radicals and then multiply both numerator and denominator by $\sqrt{6}$.

$$\sqrt{\frac{5}{6}} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{5}\cdot\sqrt{6}}{\sqrt{6}\cdot\sqrt{6}} = \frac{\sqrt{30}}{\sqrt{36}} = \boxed{\frac{\sqrt{30}}{6}}$$

---

4. $\dfrac{9}{\sqrt{27}}$

First note that $\sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3}$. Therefore, we only need to multiply the numerator and denominator by $\sqrt{3}$ to rationalize.

$$\frac{9}{\sqrt{27}} = \frac{9}{\sqrt{9}\sqrt{3}} = \frac{9}{3\sqrt{3}} = \frac{9\cdot\sqrt{3}}{3\sqrt{3}\cdot\sqrt{3}} = \frac{9\sqrt{3}}{3\sqrt{9}} = \frac{9\sqrt{3}}{9} = \boxed{\sqrt{3}}$$

5. $\dfrac{5}{\sqrt[3]{2}}$

Notice that multiplying by $\sqrt[3]{2}$ will NOT eliminate the radical from the denominator. Instead, we need to multiply by $\sqrt[3]{4}$ on both top and bottom.

$$\begin{aligned}\frac{5}{\sqrt[3]{2}} &= \frac{5\cdot\sqrt[3]{4}}{\sqrt[3]{2}\cdot\sqrt[3]{4}}\\ &= \frac{5\sqrt[3]{4}}{\sqrt[3]{8}}\\ &= \boxed{\frac{5\sqrt[3]{4}}{2}}\end{aligned}$$

---

6. $\sqrt[3]{\dfrac{4}{9}}$

As in the previous example, we will NOT eliminate the radical in the denominator by multiplying by $\sqrt[3]{9}$. Instead, we will multiply both numerator and denominator by $\sqrt[3]{3}$.

$$\begin{aligned}\sqrt[3]{\frac{4}{9}} &= \frac{\sqrt[3]{4}}{\sqrt[3]{9}}\\ &= \frac{\sqrt[3]{4}\cdot\sqrt[3]{3}}{\sqrt[3]{9}\cdot\sqrt[3]{3}}\\ &= \frac{\sqrt[3]{12}}{\sqrt[3]{27}}\\ &= \boxed{\frac{\sqrt[3]{12}}{3}}\end{aligned}$$

7. $\dfrac{2}{\sqrt{5}-3}$

To rationalize we need to multiply both numerator and denominator by $\sqrt{5}+3$ which is the conjugate of the denominator.

$$\begin{aligned}\frac{2}{\sqrt{5}-3} &= \frac{2(\sqrt{5}+3)}{(\sqrt{5}-3)(\sqrt{5}+3)}\\ &= \frac{2(\sqrt{5}+3)}{\sqrt{25}+3\sqrt{5}-3\sqrt{5}-9}\\ &= \frac{2(\sqrt{5}+3)}{5-9}\\ &= \frac{2(\sqrt{5}+3)}{-4}\\ &= \boxed{\frac{-(\sqrt{5}+3)}{2}}\end{aligned}$$

---

8. $\dfrac{1-\sqrt{2}}{\sqrt{8}+\sqrt{6}}$

To rationalize we need to multiply both numerator and denominator by $\sqrt{8}-\sqrt{6}$ which is the conjugate of the denominator.

$$\begin{aligned}\frac{1-\sqrt{2}}{\sqrt{8}+\sqrt{6}} &= \frac{(1-\sqrt{2})(\sqrt{8}-\sqrt{6})}{(\sqrt{8}+\sqrt{6})(\sqrt{8}-\sqrt{6})}\\ &= \frac{(1-\sqrt{2})(\sqrt{8}-\sqrt{6})}{\sqrt{64}+\sqrt{48}-\sqrt{48}-\sqrt{36}}\\ &= \frac{(1-\sqrt{2})(\sqrt{8}-\sqrt{6})}{8-6}\\ &= \frac{(1-\sqrt{2})(\sqrt{4}\sqrt{2}-\sqrt{6})}{2}\\ &= \boxed{\frac{(1-\sqrt{2})(2\sqrt{2}-\sqrt{6})}{2}}\end{aligned}$$

9. $\dfrac{4}{\sqrt{x} - 2\sqrt{y}}$

We will rationalize by multiplying both numerator and denominator by $\sqrt{x} + 2\sqrt{y}$.

$$\begin{aligned}
\frac{4}{\sqrt{x} - 2\sqrt{y}} &= \frac{4(\sqrt{x} + 2\sqrt{y})}{(\sqrt{x} - 2\sqrt{y})(\sqrt{x} + 2\sqrt{y})} \\
&= \frac{4(\sqrt{x} + 2\sqrt{y})}{\sqrt{x^2} - 2\sqrt{xy} + 2\sqrt{xy} - 4\sqrt{y^2}} \\
&= \boxed{\frac{4(\sqrt{x} + 2\sqrt{y})}{x - 4y}}
\end{aligned}$$

10. $\dfrac{15}{\sqrt{7} + \sqrt{2}}$

We will rationalize by multiplying the numerator and denominator by $\sqrt{7} - \sqrt{2}$.

$$\begin{aligned}
\frac{15}{\sqrt{7} + \sqrt{2}} &= \frac{15(\sqrt{7} - \sqrt{2})}{(\sqrt{7} + \sqrt{2})(\sqrt{7} - \sqrt{2})} \\
&= \frac{15(\sqrt{7} - \sqrt{2})}{\sqrt{49} + \sqrt{14} - \sqrt{14} - \sqrt{4}} \\
&= \frac{15(\sqrt{7} - \sqrt{2})}{7 - 2} \\
&= \frac{15(\sqrt{7} - \sqrt{2})}{5} \\
&= \boxed{3(\sqrt{7} - \sqrt{2})}
\end{aligned}$$