Section 9.1: Rational Numbers

- **Rational numbers:** are of the form $\frac{a}{b}$ where a and b are both integers and $b \neq 0$.

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a \text{ and } b \text{ are integers}, b \neq 0 \right\}.$$

NOTE: Every integer, whole number, and fraction is a rational number.

- Let $\frac{a}{b}$ be any rational number and n be a nonzero integer. Then

$$\frac{a}{b} = \frac{na}{nb} = \frac{an}{bn}.$$

- Let $\frac{a}{b}$ be any rational number. Then

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}.$$

- $\frac{a}{b}$ is a positive rational number when either a and b are both positive or when a and b are both negative.
- $\frac{a}{b}$ is a negative rational number when a and b have different signs (one negative and one positive).

Properties of Rational Number Addition

- **Closure Property**: Rational number + Rational number = Rational number.
- **Commutative Property**: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$.
- **Associative Property**: $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$.
- **Identity Property**: $\frac{a}{b} + 0 = \frac{a}{b} = 0 + \frac{a}{b}$.
- **Additive Inverse Property**: For every rational number $\frac{a}{b}$, there exists a unique rational number $-\frac{a}{b}$ such that

$$\frac{a}{b} + -\frac{a}{b} = 0 = -\frac{a}{b} + \frac{a}{b}.$$

$-\frac{a}{b}$ is called the **additive inverse**.

Properties of Rational Number Multiplication

- **Closure Property**: Rational number $\cdot$ Rational number $=$ Rational number.
- **Commutative Property**: $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.
- **Associative Property**: $\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}$.
- **Identity Property**: $\frac{a}{b} \cdot 1 = \frac{a}{b} = 1 \cdot \frac{a}{b}$.
- **Multiplicative Inverse Property**: For every nonzero rational number $\frac{a}{b}$, there exists a unique rational number $\frac{b}{a}$ such that

$$\frac{a}{b} \cdot \frac{b}{a} = 1 = \frac{b}{a} \cdot \frac{a}{b}.$$

  $\frac{b}{a}$ is called the **multiplicative inverse** or **reciprocal**.
- **Distributive Property**:

$$\frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$$

Multiplication of Rational Numbers: Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be any rational numbers. Then

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Cross Multiplication of Rational Number Inequality: Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers with $b > 0$ and $d > 0$. Then

$$\frac{a}{b} < \frac{c}{d} \qquad \text{if and only if} \qquad ad < bc.$$

NOTE: BE CAREFUL!!! Both denominators must be positive in order to use this. DO NOT use if one of the denominators is negative unless you first the rational number using $\frac{a}{-b} = \frac{-a}{b}$.

Example 1: Put the appropriate sign (<, =, >) between each pair of rational numbers to make a true statement.

(a) $-\frac{1}{3} \qquad \frac{5}{4}$

(b) $\frac{-5}{6} \qquad \frac{-11}{12}$

(c) $\frac{-12}{15} \qquad \frac{36}{-45}$

(d) $-\frac{3}{12} \qquad \frac{4}{-20}$

Example 2: Consider the following sets:

$$\begin{aligned} W &= \text{the set of whole numbers} \\ F &= \text{the set of nonnegative fractions} \\ I &= \text{the set of integers} \\ N &= \text{the set of negative integers} \\ Q &= \text{the set of rational numbers} \end{aligned}$$

List all the sets that have the following properties.

(a) The set is closed under addition.

(b) The set is closed under subtraction.

(c) The set has the additive inverse property.

Division of Rational Numbers: Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers where $\frac{c}{d}$ is nonzero. Then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

NOTE: All the operations that we performed with fractions can be extended to rational numbers.