Section 9.2: Real Numbers

- **Irrational Numbers**: Set of numbers which have an infinite non-repeating decimal approximation. An irrational number cannot be written as a fraction.
- **Real Numbers**: are the union of rational and irrational numbers.
- **Square root**: Let a be a nonnegative real number. Then the **square root** of a (**principal square root**), denoted $\sqrt{a}$, is defined as

$$\sqrt{a} = b \quad \text{where} \quad b^2 = a, \text{ and } \; b \geq 0.$$

Example 1: Simplify the following square roots.

(a) $\sqrt{16}$

(b) $\sqrt{169}$

(c) $\sqrt{49}$

(d) $\sqrt{81}$

- **n-th roots**: Let a be a real number and n be a positive integer. For $\sqrt[n]{a}$, a is called the **radicand** and n is called the **index**.
  1. If $a \geq 0$, then $\sqrt[n]{a} = b$ if and only if $b^n = a$ and $b \geq 0$.
  2. If $a < 0$ and n is odd, then $\sqrt[n]{a} = b$ if and only if $b^n = a$.
  3. If $a < 0$ and n is even, then $\sqrt[n]{a}$ is undefined.

Example 2: Simplify the following n-roots, if possible.

(a) $\sqrt[4]{16}$

(b) $\sqrt[3]{-27}$

(c) $\sqrt[3]{125}$

(d) $\sqrt[5]{32}$

(e) $\sqrt[4]{81}$

(f) $\sqrt[4]{-81}$

Rules for n-th roots:

- **Product rule for radicals**: If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a positive integer,

$$\boxed{\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.}$$

  In other words, the product of radicals is the radical of the product.

- **Quotient rule for radicals**: If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a positive integer,

$$\boxed{\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.}$$

  In other words, the radical of a quotient is the quotient of the radicals.

- If n is even, then $\sqrt[n]{a^n} = |a|$. For example, $\sqrt[4]{(-2)^4} = |-2| = 2$.

- If n is odd, then $\sqrt[n]{a^n} = a$. For example, $\sqrt[3]{(-6)^3} = -6$.

Simplifying radicals: A radical is in simplest form when the following conditions are satisfied.

- The quantity under the radical has no factor raised to a power greater than or equal to the index.

- There is no fraction under the radical.

- There is no radical in the denominator.

- There is no common factor, other than 1, between the exponents on factors under the radical and the index.

Example 3: Simplify the following radicals.

(a) $\sqrt{44}$

(b) $\sqrt{75}$

(c) $-\sqrt{24}$

(d) $\sqrt{242}$

(e) $\sqrt[3]{-250}$

(f) $\sqrt[4]{48}$

(g) $3\sqrt{250}$

(h) $-2\sqrt{63}$

(i) $7\sqrt[4]{32}$

(j) $5\sqrt[4]{243}$

Example 4: Add or subtract, if possible. Write all answers in simplest form.

(a) $5\sqrt{10} + 8\sqrt{10} - 3\sqrt{10}$

(b) $\sqrt{18} + \sqrt{98}$

(c) $3\sqrt{48} - \sqrt{24}$

(d) $5\sqrt{12} - \sqrt{75} + 2\sqrt{20}$

(e) $9\sqrt{7} - 5\sqrt{72} + 4\sqrt{18} - 5\sqrt{112}$

Example 5: Multiply. Write all answers in simplest form.

(a) $\sqrt{3} \cdot \sqrt{12}$

(b) $\sqrt{2}\left(\sqrt{6} - 5\right)$

(c) $\left(\sqrt{2} - 4\right)\left(\sqrt{2} + 5\right)$

(d) $\left(3+\sqrt{8}\right)\left(2-\sqrt{2}\right)$

(e) $\left(\sqrt{3}+5\right)\left(\sqrt{3}-4\sqrt{2}\right)$

Example 6: Divide. Write all answers in simplest form.

(a) $\dfrac{4}{\sqrt{6}}$

(b) $\dfrac{\sqrt{3}}{\sqrt{5}}$

(c) $\dfrac{3\sqrt{2}}{5\sqrt{3}}$

(d) $\dfrac{2\sqrt{3}}{4\sqrt{12}}$

- **Rational exponents and Radicals**: Let a be any real number and $\frac{m}{n}$ be a rational number is simplest form. Then

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

NOTE: In a rational exponent, the numerator indicates the power and the denominator indicates the root.

Example 7: Simplify each expression if possible.

(a) $8^{2/3} =$

(e) $(-125)^{-2/3} =$

(b) $16^{3/4} =$

(f) $81^{-3/4} =$

(c) $27^{1/3} =$

(g) $-25^{3/2} =$

(d) $(-64)^{2/3} =$

(h) $(-16)^{3/4} =$

NOTE: The properties that held for integer exponents also hold for rational exponents.

Solving Inequalities

- **Addition Property of Inequality:** If a, b, and c are real numbers, then

  $$a < b \quad \text{and} \quad a + c < b + c$$

  are equivalent. (That is, you can add or subtract the same quantity on both sides of the inequality without changing the solution.)

- **Multiplication Property of Inequality:** For all real numbers a, b, and c, with $c \neq 0$,

  1. $a < b$ and $ac < bc$ are equivalent if $c > 0$.

  2. $a < b$ and $ac > bc$ are equivalent if $c < 0$.

  (That is, whenever you multiply or divide by a negative number you must reverse or flip the inequality.)

Example 8: Solve the following inequalities

(a) $3x + 5 \geq 6x - 7$

(c) $x - \dfrac{2}{3} > \dfrac{5}{6}$

(b) $-2x + 4 \leq 11$

(d) $\dfrac{3}{2}x - 3 < \dfrac{5}{6}x + \dfrac{1}{3}$