MATH 11008: Euler’s Theorems
Section 5.5

- **Euler path**: An *Euler path* in a connected graph is a path that travels through ALL the edges of the graph exactly once.

- **Euler circuit**: An *Euler circuit* in a connected graph is a circuit that travels through every edges exactly once.
  
  ○ A connected graph cannot have both an Euler path and an Euler circuit. It can have one or the other or neither.
  
  ○ If a graph is disconnected, then an Euler circuit is impossible.

**Euler’s Circuit Theorem:**

1. If a graph is connected and every vertex is even, then it has an Euler circuit (at least one, usually more).

2. If a graph has any odd vertices, then it does not have an Euler circuit.

**Example 1:** Determine if the following graph has an Euler circuit. Explain why or why not.
**Euler’s Path Theorem:**

1. If a graph is connected and has exactly two odd vertices, then it has an Euler path (at least one, usually more). Any such path must start at one of the odd vertices and end at the other.

2. If a graph has more than two odd vertices, then it cannot have an Euler path.

**Example 2:** Determine if the following graph has an Euler path. Explain why or why not.

![Graph Diagram]

**Euler’s Sum of Degrees Theorem:**

1. The sum of the degrees of all the vertices of a graph equals twice the number of edges (and therefore is an even number).

2. A graph always has an even number of odd vertices.

**Example 3:** Determine the number of edges in a graph with 8 vertices, two of degree 1, three of degree 2, two of degree four, and one of degree 6.
The following table summarizes the previous two theorems for a connected graph $G$. Remember that if a graph is disconnected, it cannot have an Euler path nor an Euler circuit.

**Summary of Euler’s Theorems (Assuming $G$ is connected)**

<table>
<thead>
<tr>
<th>Number of odd vertices</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$G$ has an Euler circuit.</td>
</tr>
<tr>
<td>2</td>
<td>$G$ has an Euler path.</td>
</tr>
<tr>
<td>4, 6, 8…</td>
<td>$G$ has neither.</td>
</tr>
<tr>
<td>1, 3, 5, …</td>
<td>This is impossible.</td>
</tr>
</tbody>
</table>