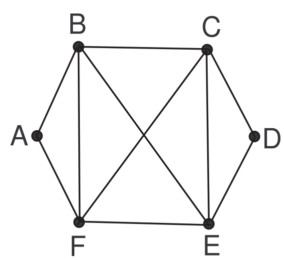
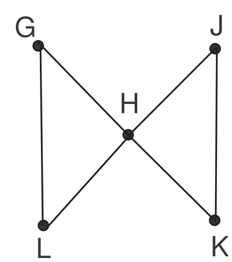
MATH 11008: Hamilton Path and Circuits
Sections 6.1, 6.2 & 6.3

- **Hamilton Path:** A **Hamilton path** is a path in a graph that includes each vertex of the graph once and only once.
- **Hamilton Circuit:** A **Hamilton circuit** is a circuit that visits each vertex exactly once (returning to the starting vertex to complete the circuit).
  - Note the difference: Euler paths/circuits cover all edges only once and Hamilton paths/circuits cover all vertices only once.
  - If a graph has a Hamilton circuit, then it automatically has a Hamilton path.
  - A graph can have a Hamilton path but not have a Hamilton circuit.
  - The existence of an Euler path or Euler circuit tell us nothing about the existence of a Hamilton path or Hamilton circuit.
  - Although Euler's Theorems tell us about when a graph has an Euler path or circuit and when it does not, no analogous theorems about Hamilton paths or circuits exist. In other words, there is no convenient theorem that gives necessary and sufficient conditions for a Hamilton circuit to exist. Therefore, to determine if a graph has a Hamilton path or Hamilton circuit, you just need to try to find one.

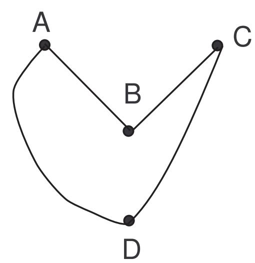
Example 1: For each graph, give an example of a Hamilton circuit, if possible.

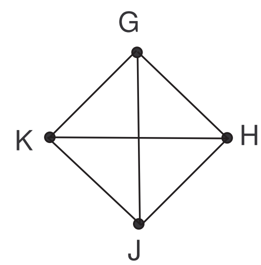




- **Complete Graph:** A graph with N vertices in which every pair of distinct vertices is joined by an edge is called a **complete graph** on N vertices and denoted by the symbol K_N.
  - Note that in a complete graph K_N every vertex has degree $N-1$.
  - K_N has $\dfrac{N(N-1)}{2}$ edges.

Example 2: Determine if the following are complete graphs.





- **Hamilton circuits for complete graphs:** Any complete graph with three or more vertices has a Hamilton circuit.
- **When Hamilton circuits are the same:** Hamilton circuits that differ only in their starting points will be considered to be the same circuit. For example, in the first graph of Example 1,
$$A, B, C, D, E, F, A \quad \text{and} \quad C, D, E, F, A, B, C$$
are the same Hamilton circuit, since only their starting points differ.
- **Number of Hamilton Circuits in a complete graph:** A complete graph with N vertices has $(N-1)!$ Hamilton circuits.

Example 3: For a complete graph with 4 vertices, how many Hamilton circuits does it have?

Example 4: Short answer.

(a) How many edges are there in K_{200}?

(b) How many edges are there in K_{201}?

(c) If the number of edges in K_{500} is x, and the number of edges in K_{501} is y, what is the value of $y - x$?

Example 5: In each case, find the value of N.

(a) K_N has 720 distinct Hamilton circuits.

(b) K_N has 66 edges.

(c) K_N has 80,200 edges.

- **Traveling Salesman Problem:** Any problem that has a traveler, a set of sites, a cost function for travel between sites (weights on the edges), and need to tour all the sites (Hamilton circuit), and a desire to minimize the total cost of the tour (Hamilton circuit of least total weight) is known as a **traveling salesman problem** (TSP).
- **total weight of a circuit:** The **total weight** of a circuit is the sum of the weights on the edges of the circuit.
- **Minimum Hamilton circuit:** In a weighted graph, a **minimum Hamilton circuit** is a Hamilton circuit with smallest possible total weight.