MATH 11009: Exponential & Logarithmic Equations Section 5.3

- **Exponential Function:** If $b > 0,\ b \neq 1$, then the function $f(x) = b^x$ is an exponential function. The constant b is called the **base** of the function and the variable x is the **exponent**.

- **Logarithmic Function:** For $x > 0,\ b > 0$, and $b \neq 1$, the logarithmic function to base b is $y = \log_b x$ which is defined by $x = b^y$.

  * Remember $\ln x$ is a logarithmic function with base e.
  * Remember $\log x$ is a logarithmic function with base 10.

- **Basic Properties of Logarithms:** For $b > 0,\ b \neq 1$, we have

  ◦ $\log_b b = 1$
  ◦ $\log_b b^x = x$
  ◦ $\log_b 1 = 0$
  ◦ $b^{\log_b x} = x$

- **Logarithmic Properties using base 10 and base e.**

  $\log 10 = 1$ $\qquad$ $\ln e = 1$
  $\log 1 = 0$ $\qquad$ $\ln 1 = 0$
  $\log 10^x = x$ $\qquad$ $\ln e^x = x$
  $10^{\log x} = x$ $\qquad$ $e^{\ln x} = x$

Example 1. Use the properties of logarithms to evaluate the following.

(a) $\ln e^5$

(b) $7^{\log_7 24}$

(c) $\log 10^{-4}$

- **Laws of Logarithms:** Let $b > 0$, $b \neq 1$, n a real number, and M and N be positive real numbers
  - $\log_b(MN) = \log_b M + \log_b N$
  - $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
  - $\log_b M^n = n \log_b M$

Example 2. Rewrite each expression as the sum, difference, or product of logarithms, and simplify where possible.

(a) $\log \dfrac{\sqrt{x+2}}{4x}$

(b) $\log_3 \left(\dfrac{x^2(x-5)^8}{(3x+2)^4}\right)$

Example 3. Rewrite each expression as a single logarithm.

(a) $3\ln x - 5\ln(x+2) + 7\ln(x-9)$

(b) $4\log x - \frac{1}{2}\left[\log(x+1) + \log(4x-5)\right]$

- **Change of Base Formula:** If $b > 0,\ b \neq 1,\ a > 0,\ a \neq 1$, and $x > 0$, then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular, for $b = 10$ and $b = e$,

$$\log_a x = \frac{\log x}{\log a} \qquad \text{and} \qquad \log_a x = \frac{\ln x}{\ln a}$$

Example 4. Use the change of base formula to calculate $\log_9 11$ to four decimal places.

- **Guidelines for solving exponential equations**:
  1. Isolate the exponential expression on one side of the equation.
  2. Take the logarithm of each side in order to "bring down" the variable in the exponent. (It does not matter which base you use on the logarithm as long as it is the same on both sides.)
  3. Solve for the variable.

Example 5. Solve $6e^x = 30$

Example 6. Solve $4(5^{3x}) = 16$

Example 7. Solve $3\left(2 + 10^{4x}\right) = 18$

- **Guidelines for solving logarithmic equations**:

  1. Isolate the logarithmic term on one side of the equation. This is accomplished by using the laws of logarithms.
  2. Rewrite the equation in exponential form.
  3. Solve for the variable.
  4. Check to make sure you don't have extraneous solutions. To do this, substitute "answers" into the original equation and check that you are not taking the logarithm of a negative number or zero.

Example 8. Solve $\log x + \log(x - 15) = 2$

Example 9. Solve $\ln x - 6 = \ln(x + 4)$

Example 10. The demand function for a dining room table is given by $p = 4000(3^{-q})$ dollars per table, where p is the price and q is the quantity, in thousands of tables, demanded at that price. What quantity will be demanded if the price per table is $256.60?

Example 11. The population in a certain city was 53,000 in 2000, and its future size is predicted to be $P = 53,000e^{0.015t}$ people, where t is the number of years after 2000. Determine when the population will reach 60,000.

SUPPLEMENTAL EXERCISES

Solve for x.

1. $\log(7x+2)=1$
2. $2\log(x+3)=4$
3. $3\log(2x+1)-2=1$
4. $\log x-\log(3x-2)=2$
5. $5\ln(2x-3)+2=2$
6. $\ln(2x-3)-\ln(x-1)=0$
7. $\ln(3x-2)=\ln(2x+5)$
8. $2\ln x-\ln(x+12)=0$
9. $\log x+\log(x-3)=1$
10. $\ln(8x-3)=\ln(4x+1)$
11. $\log(x-1)+\log(x-3)=0$
12. $\ln x+\ln(x+3)=\ln(x+8)$
13. $\log(x+3)+\log(x+1)=\log(x+7)$
14. $\log 3x-\log(x-2)=1$
15. $\ln(x^2-12)-\ln(3x-2)=0$
16. $\ln(x^2-8)-\ln(x+4)=0$

ANSWERS

1. $x = 8/7$
2. $x = 97$
3. $x = 9/2$
4. $x = 200/299$
5. $x = 2$
6. $x = 2$
7. $x = 7$
8. $x = 4$
9. $x = 5$
10. $x = 1$
11. $x = 2 + \sqrt{2}$
12. $x = 2$
13. $x = 1$
14. $x = 20/7$
15. $x = 5$
16. $x = 4, \;\; -3$