MATH 11009: Rational Functions Section 6.5

- **Rational Function:** The function f is a rational function if
$$f(x) = \frac{P(x)}{Q(x)}$$
where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

- **Vertical Asymptotes:** A vertical asymptote occurs in the graph of $f(x) = \dfrac{P(x)}{Q(x)}$ at those values of x where $Q(x) = 0$ and $P(x) \neq 0$; that is, at values of x where the denominator equals zero but the numerator does not equal zero.

> **Finding Vertical Asymptotes:**
> To find where the vertical asymptotes occur, set the denominator equal to zero and solve for x. If any of these values of x do not also make the numerator equal to zero, a vertical asymptote occurs at those values.

- **Horizontal Asymptotes:** If y approaches a as x approaches ∞ or $-\infty$, the graph of $y = f(x)$ has a horizontal asymptote at $y = a$.

> **Finding Horizontal Asymptotes:** Consider $f(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomials.
>
> - **Degree of numerator = Degree on denominator:** When the numerator and denominator of a rational function have the same degree, the line $y = \dfrac{a}{b}$ is the horizontal asymptote, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.
> - **Degree of numerator < Degree of denominator:** When the degree of the numerator is less than the degree of the denominator, the line $y = 0$ (x-axis) is the horizontal asymptote.
> - **Degree of numerator > Degree of denominator:** When the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

Example 1. For each of the following rational functions, identify the vertical asymptote(s) and horizontal asymptote, if they exist.

(a) $f(x) = \dfrac{x-7}{x+2}$

(b) $g(x) = \dfrac{5x^2-4x+3}{6x^2-5x+1}$

(b) $h(x) = \dfrac{7x-9}{4x^2-4x-15}$

(d) $k(x) = \dfrac{x^3-8}{x^2-36}$

- **Rational Equation:** A rational equation is an equation involving rational functions.
- **Solving rational equations analytically:**
  1. Multiply both sides of the equation by the LCD of the fractions in the equation.
  2. Solve the resulting polynomial equation for the variable.
  3. Check each solution in the original equation. Some solutions to the polynomial equation may not be solutions to the original rational equation; these are called **extraneous solutions**.

Example 2. Solve: $\dfrac{2x-3}{2x-7} = \dfrac{x}{2x+3}$

Example 3. Solve $\dfrac{1}{x} + \dfrac{1}{2} = 3$

Example 4. Solve $\dfrac{1}{x-4} + \dfrac{x}{x+4} = \dfrac{14}{x^2-16}$

Example 5. Solve $\dfrac{2}{3x+1} + \dfrac{6x}{3x+1} = \dfrac{1}{x}$

EXERCISES

Solve for x:

1. $\dfrac{3x}{x-1} - \dfrac{2x}{x+2} = 0$

2. $\dfrac{5x}{x^2-4} - \dfrac{x}{x-2} = 0$

3. $\dfrac{4}{x+3} - \dfrac{5}{x-2} = \dfrac{2}{x-2}$

4. $\dfrac{3x}{x^2-9} - \dfrac{2}{x+3} = \dfrac{5}{x-3}$

5. $\dfrac{5x-2}{7x+8} = 3$

6. $\dfrac{2x}{x-2} + 5 = \dfrac{7}{x-2}$

7. $\dfrac{2x^2-4x-1}{x^2} = 1$

8. $\dfrac{2x^2-x-6}{x^2+9} = 0$

9. $\dfrac{2x^2-3x-6}{x^2-4} = 0$

10. $\dfrac{4}{x+3} + \dfrac{2x}{x-4} = 0$

11. $\dfrac{x+3}{2x-1} = \dfrac{x-4}{x+1}$

12. $\dfrac{3x}{x-1} = \dfrac{x+2}{x+1}$

13. $\dfrac{9x-1}{x+3} - 5 = 0$

14. $\dfrac{2x}{x-1} + \dfrac{3}{x} = 1$

15. $\dfrac{2x}{x-5} = \dfrac{x}{x+3}$

16. $\dfrac{2x+5}{x-7} = \dfrac{1}{5}$

17. $\dfrac{4x}{x-5} + \dfrac{1}{x-2} = 0$

18. $\dfrac{x^2-6x+6}{3x-7} = 0$

19. $\dfrac{3x-2}{x+8} = \dfrac{2}{3}$

20. $\dfrac{4x^2-x-1}{x} = 1$

ANSWERS

1. $x = 0, \ \ -8$
2. $x = 0, \ \ 3$
3. $x = -29/3$
4. $x = -9/4$
5. $x = -13/8$
6. $x = 17/7$
7. $x = 2 \pm \sqrt{5}$
8. $x = 2, \ \ -3/2$
9. $x = \dfrac{3}{4} \pm \dfrac{\sqrt{57}}{4}$
10. $x = -\dfrac{5}{2} \pm \dfrac{\sqrt{57}}{2}$
11. $x = \dfrac{13}{2} \pm \dfrac{\sqrt{165}}{2}$
12. No solution
13. $x = 4$
14. $x = -2 \pm \sqrt{7}$
15. $x = 0, \ \ -11$
16. $x = -32/9$
17. $x = \dfrac{7}{8} \pm \dfrac{\sqrt{129}}{8}$
18. $x = 3 \pm \sqrt{3}$
19. $x = 22/7$
20. $x = \dfrac{1}{4} \pm \dfrac{\sqrt{5}}{4}$