MATH 11009: Transformations
Section 4.1

- **Vertical Shift**: Suppose that $k > 0$.
  - The equation $y = f(x) + k$ shifts the graph of $f(x)$ up k units. (Adding a constant on the outside of a functions shifts the graph up.)
  - The equation $y = f(x) - k$ shifts the graph of $f(x)$ down k units. (Subtracting a constant on the outside of a function shifts the graph down.)

- **Horizontal Shift**: Suppose that $h > 0$.
  - The equation $y = f(x + h)$ shifts the graph of $f(x)$ to the left h units. (Adding a constant inside the function shifts the graph left.)
  - The equation $y = f(x - h)$ shifts the graph of $f(x)$ to the right h units. (Subtracting a constant inside the function shifts the graph right.)

- **Reflections**:
  - The equation $y = -f(x)$ reflects the graph of $f(x)$ with respect to the $x-$axis. (Multiplying by a negative on the outside of a function flips the graph with respect to the $x-$axis.)
  - The equation $y = f(-x)$ reflects the graph of $f(x)$ with respect to the the $y-$axis. (Multiplying by a negative inside the function flips the graph with respect to the $y-$axis.)

- **Vertical Stretching and Compression**:
  - When $|a| > 1$, the equation $y = af(x)$ stretches the graph of $f(x)$ vertically by a factor of $|a|$. (Multiplying by a number, larger than one in absolute value, on the outside of a function causes the graph to be stretched by a factor of $|a|$.)
  - When $|a| < 1$, the equation $y = af(x)$ compresses the graph of $f(x)$ vertically by a factor of $|a|$. (Multiplying by a number, less than one in absolute value, on the outside of a functions causes the graph to compress or shrink by a factor of $|a|$.)

Example 1. Explain how the graph of g is obtained from the graph of f. Be specific!

(a) $f(x) = x^2; \qquad g(x) = (x-4)^2$

(b) $f(x) = x^2; \qquad g(x) = (x+7)^2 - 5$

(c) $f(x) = \sqrt{x}; \qquad g(x) = 3\sqrt{x+1}$

(d) $f(x) = x^3; \qquad g(x) = \frac{1}{4}(x-5)^3 + 2$

(e) $f(x) = \sqrt[3]{x}; \qquad g(x) = -4\sqrt[3]{x-2} + 6$

Example 2. Suppose the graph of $y = x^{3/2}$ is shifted to the right 4 units, reflected about the $x-$axis, and shifted down 5 units. What is the equation that gives the new graph?

Example 3. A function f is given, and the indicated transformations are applied to its graph in the given order. Write the equation for the final transformed graph.

(a) $f(x) = \sqrt{x}$; reflected about the $y-$axis, vertically compressed by a factor of $\dfrac{1}{9}$, vertical shift down 8 units.

(b) $f(x) = x^2$; reflected about the $x-$axis, vertically stretched by a factor of 8, horizontal shift left 5 units

Example 4. The number of U.S. aircraft accidents, for all military services, can be modeled by $y = 0.184x^2 - 5.437x + 58.427$, with $x = 0$ in 1970. Rewrite the model with x equal to the number of years from 1960.

Algebraic Tests for Symmetry:

- **Symmetric with respect to the x-axis:** If replacing y with $-y$ produces an equivalent equation, then the graph is **symmetric with respect to the x-axis**.
- **Symmetric with respect to the y-axis:** If replacing x with $-x$ produces an equivalent equation, then the graph is **symmetric with respect to the y-axis**.
- **Symmetric with respect to the origin:** If replacing x with $-x$ AND y with $-y$ produces an equivalent equation, then the graph is **symmetric with respect to the origin**.

Example 7. Determine algebraically whether the graph of the given equation is symmetric with respect to the x-axis, the y-axis, and/or the origin.

(a) $y = -x^2 + 4$

(b) $y = -x^3 + 5x$

Even and Odd Functions:

- **Even Function:** If the graph of a function f is symmetric with respect to the y-axis, we say that it is an **even function**. That is, for each x in the domain of f, $f(-x) = f(x)$.
- **Odd Function:** If the graph of a function f is symmetric with respect to the origin, we say that it is an **odd function**. That is, for each x in the domain of f, $f(-x) = -f(x)$.

Example 6. Determine if the following functions are even, odd, or neither.

(a) $f(x) = 7x^3 - 5x$

(c) $h(x) = 2x^2 + x + 1$

(b) $g(x) = 3x^2 + 4$

(d) $k(x) = |x + 7|$