MATH 11010: Applications Section 4.6

- **Exponential functions:** The function $f(x) = a^x$, where x is a real number, $a > 0$ and $a \neq 1$, is called an **exponential function** with base a.

- **Compound Interest**: The amount of money A that a principal P will grow to after t years at interest rate r (in decimal form), compounded n times per year, is given by the formula:

$$\boxed{A = P\left(1 + \frac{r}{n}\right)^{nt}}$$

Example 1: Suppose \$5,000 is invested in an account earning 3.7% interest, compounded monthly. How long will it take the account to reach \$9,000?

- **Compounded Continuously Interest**: Suppose that an amount P is invested in a savings account at an interest rate of r (written as a decimal), *compounded continuously.* The amount A in the account after t years is given by

$$\boxed{A = Pe^{rt}}$$

Example 2: Suppose that \$12,000 is invested in an account paying 5.6% interest compounded continuously.

(a) Find the exponential equation that describes the amount in the account after time t, in years.

(b) What is the balance after 3 years?

(c) How long does it take the account to double?

- **Exponential Growth**: Suppose that the initial population is P_0 and the exponential growth rate is k (written as a decimal). Then the population $P(t)$ after time t is given by:

$$\boxed{P(t) = P_0 e^{kt}}$$

- **Growth Rate and Doubling Time:** The growth rate k and the doubling time T are related by

$$kT = \ln 2, \qquad \text{or} \qquad k = \frac{\ln 2}{T}, \qquad \text{or} \qquad T = \frac{\ln 2}{k}$$

Example 3: Under ideal conditions, a population of rabbits has an exponential growth rate of 11.7% per day. Consider an initial population of 100 rabbits.

(a) Find the exponential growth function.

(b) What will the population be after 7 days?

(c) What will the population be after 2 weeks?

(d) Find the doubling time.

(e) How long will it take for the rabbit population to reach 315 rabbits? Round answer to two decimal places.

- **Exponential Decay**: Suppose that P_0 represents the amount of a substance at time $t = 0$, and k is a decay rate (positive constant) that depends on the situation. Then $A(t)$, the amount of the substance left after time t, can be found using the following formula:

$$\boxed{A(t) = P_0 e^{-kt}}$$

Example 4: What is the relationship between the decay constant k and the half-life?

Example 5: The half-life of radium-226 is 1590 years. Suppose we begin with a 150 mg sample.

(a) Find the decay constant k.

(b) Find a function that models the mass remaining after t years.

(c) Find the mass that will remain after 900 years? (Round to two decimal places)

(d) After how many years will only 50 mg of the sample remain? (Round to two decimal places)

Homework: pp 419–420; #1, 3, 7, 11, and supplemental exercises

SUPPLEMENTAL EXERCISES

1. If $\$22,450$ is invested in an account earning $4\frac{1}{2}\%$ per year compounded quarterly, determine the amount in the account at the end of 5 years. (Round answers to two decimal places.)

2. A couple just had a new child. How much should they invest now at 8.25%, compounded continuously, in order to have $\$80,000$ for the child's education 17 years from now? (Round answers to two decimal places.)

3. An adoring aunt wants to guarantee that her niece will have enough money for college. How much should the aunt deposit into an account today with annual interest rate 4.97%, compounded quarterly, in order for the niece to have $\$45,000$ for college 18 years from now? (Round answers to two decimal places.)

4. The relative growth rate for a certain bacteria population is 47% per day. A small culture is formed and at the end of 5 days the count showed approximately $45,870$ bacteria in the culture.

   (a) Find the initial number of bacteria in this culture. (Round answer to the nearest whole unit.)

   (b) Find the projected number of bacteria 8 days after the culture was formed. (Round answer to the nearest whole unit).

5. The turtle population in local lake grows exponentially. The current population is 54 turtles and the relative growth rate is 21% per year.

   (a) Find a function that models the turtle population after t years.

   (b) Find the projected turtle population after 7 years. (Round to nearest whole unit).

6. The half-life of Osiki-11010 is 24 years. Suppose we have a 30g sample.

   (a) Find the function that models the mass remaining after t years.

   (b) How much of the sample will remain after 35 years? (Round to two decimal places)

7. Determine the principal that must be invested at a rate of $7\frac{1}{2}\%$, compounded monthly, so that $\$500,000$ will be available for retirement in 20 years. (Round to two decimal places)

8. The relative growth rate for a certain bacteria population is 115% per hour. Three hours after the culture is formed, a count shows approximately 13,500 bacteria.
   (a) Find the initial number of bacteria in the culture. (Round to nearest whole unit).
   (b) Estimate the number of bacteria 5 hours from the time the culture was started. (Round to nearest whole unit).

9. The stray-cat population in a small town grows exponentially. In 1999, the town had 30 stray cats and the relative growth rate was 15% per year.
   (a) Find a function that models the stray-cat population t years after 1999.
   (b) Find the projected population in 2011. (Round to nearest whole unit).

10. A sum of $\$5,000$ is invested at an interest rate of $8\frac{1}{2}\%$ per year, compounded quarterly. Find the amount of the investment after 2 years. (Round to two decimal places).

11. Under ideal conditions, a certain type of bacteria has a relative growth rate of 185% per hour. A number of these bacteria are introduced into a food product. Four hours after contamination, a bacterium count shows that there are about $132,000$ bacteria in the food.
    (a) Find the initial number of bacteria introduced into the food. (Round to nearest whole unit).
    (b) Estimate the number of bacteria in the food six hours after contamination. (Round to nearest whole unit).

12. The half-life of cesium-137 is 30 years. Suppose we have a 10 g sample.
    (a) Find the function that models the mass remaining after t years.
    (b) How much of the sample will remain after 40 years? (Round to two decimal places)

13. The amount (in milligrams) of a drug in the body t hours after taking a pill is given by $A(t) = 25\,(0.85)^t$.
    (a) What is the initial dose given?
    (b) What is the amount of the drug left after 10 hours?

14. An investment of \$2000 is made at 3.9%, compounded monthly. How long will it take for the investment to grow to \$12000?

15. An investment of \$4000 is made at 7.21%, compounded quarterly. How long will it take for the investment to double?

16. An investment of \$400 is made at 2.25%, compounded continuously. How long will it take for the investment to double?

17. An investment grows at 4.23% compounded monthly. How many years will it take for the investment to increase by 75%?

18. An amount of money doubles at the end of 11 years when invested at a certain interest rate that is compounded continuously. What is the rate of interest?

19. The turtle population in local lake grows exponentially. The current population is 54 turtles and the relative growth rate is 21% per year. Find the number of years required for the turtle population to reach 500.

20. The half-life of a radioactive isotope is 24 years. Suppose we begin with a 30g sample.

    (a) Find the function that models the amount of material remaining after t years.

    (b) After how long will only 8 g of the sample remain?

21. The half-life of math-17 is 415 years. Suppose we begin with a 90 g sample.

    (a) Find a function that models the mass remaining after t years.

    (b) Find the mass that will remain after 275 years? (Round to two decimal places)

    (c) After how many years will only 15 g of the sample remain? (Round answer to two decimal places.)

ANSWERS

1. $\$28,079.35$

2. $\$19,678.50$

3. $\$18,496.37$

4. (a) $4,375$ bacteria
   (b) $187,899$ bacteria

5. (a) $P = 54e^{.21t}$
   (b) 235 turtles

6. (a) $A(t) = 30e^{-0.0289t}$
   (b) 10.91 g

7. $\$112,087.09$

8. (a) 429 bacteria
   (b) $134,788$ bacteria

9. (a) $P = 30e^{0.15t}$
   (b) 181 cats

10. $\$5915.98$

11. (a) 81 bacteria
    (b) $5,359,864$

12. (a) $A(t) = 10e^{-.0231t}$
    (b) 3.97 g

13. (a) 25
    (b) 4.92 mg

14. 46.02 years

15. 9.70 years

16. 30.81 years

17. 13.25 years

18. 6.30%

19. 10.60 years

20. (a) $A = 30e^{-0.0289t}$
    (b) 45.77 years

21. (a) $M = 90e^{-t\frac{ln2}{415}}$
    (b) 56.85 g
    (c) 1072.76 yrs