MATH 11010: Circle, Ellipses, Hyperbolas Sections 6.2 & 6.3

- **Circle:** A **circle** is the set of all points that are a fixed distance (the **radius**) from a fixed point (the **center**) in the plane.
- The standard equation of a circle with center (h, k) and radius r is given by

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

- If the center is the origin $(0, 0)$, then the equation of the circle with radius r is

$$x^2 + y^2 = r^2.$$

- The **unit circle** is the circle with center $(0, 0)$ and radius 1. Its equation is

$$x^2 + y^2 = 1.$$

Example 1. Identify the center and radius of $(x-3)^2 + (y+5)^2 = 16$.

Example 2. Find the equation of the circle having center at $(-4, 5)$ and radius 7.

Example 3. Find the center and the radius of the circle with the given equation.

$$x^2 + y^2 - 2x + 6y + 1 = 0.$$

- **Ellipse:** An **ellipse** is the set of all points in a plane, the sum of whose distances from two fixed points (the **foci**) is a constant. The **center** of an ellipse is the midpoint of the segment between the foci.

Standard Equation of an Ellipse with Center at the Origin

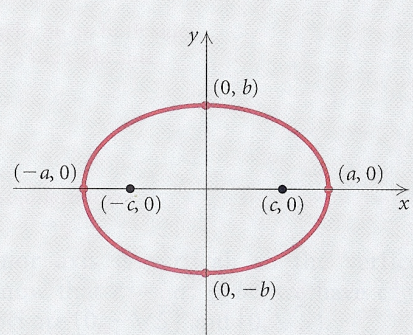
Major Axis Horizontal

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0$$

Vertices: $(-a, 0), (a, 0)$

y-intercepts: $(0, -b), (0, b)$

Foci: $(-c, 0), (c, 0)$, where $c^2 = a^2 - b^2$



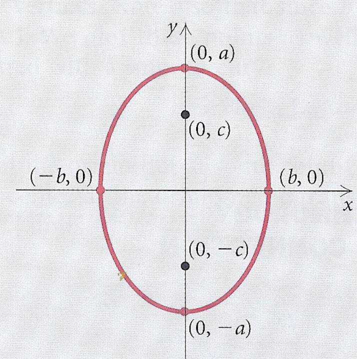
Major Axis Vertical

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b > 0$$

Vertices: $(0, -a), (0, a)$

x-intercepts: $(-b, 0), (b, 0)$

Foci: $(0, -c), (0, c)$, where $c^2 = a^2 - b^2$



Example 4. Find the vertices and the foci of the ellipse with the equation:

$$\frac{x^2}{25} + \frac{y^2}{36} = 1.$$

Example 5. Find the vertices and the foci of the ellipse with the equation:

$$5x^2 + 7y^2 = 35.$$

Example 6. Find an equation of an ellipse satisfying the given conditions:

Vertices: $(0, -6)$ and $(0, 6)$;
foci: $(0, -4)$ and $(0, 4)$

Standard Equation of an Ellipse with Center at (h, k)

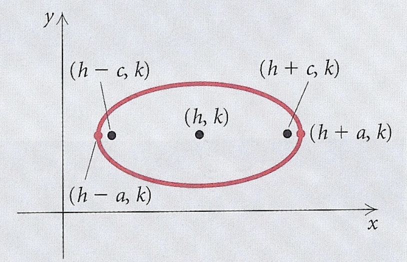
Major Axis Horizontal

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \ a > b > 0$$

Vertices: $(h-a, k), (h+a, k)$

Length of minor axis: $2b$

Foci: $(h-c, k), (h+c, k)$, where $c^2 = a^2 - b^2$



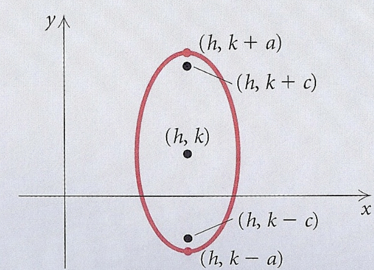
Major Axis Vertical

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \ a > b > 0$$

Vertices: $(h, k-a), (h, k+a)$

Length of minor axis: $2b$

Foci: $(h, k-c), (h, k+c)$, where $c^2 = a^2 - b^2$



- Note that if $a = b$, then

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1; \qquad \text{clear the denominators}$$

$$(x-h)^2 + (y-k)^2 = a^2.$$

But this is the equation of a circle with center (h, k) and radius a. Hence, when $a = b$, the equation is actually a circle and not an ellipse.

- **Hyperbola:** A **hyperbola** is the set of all points in a plane for which the absolute value of the difference of the distance from two fixed points (the **foci**) is constant. The midpoint of the segment between the foci is the **center** of the hyperbola.
  - The line segment connecting the two vertices is called the **transverse axis**.
  - The asymptotes of the hyperbola are given by
    - $y = -\frac{b}{a}x$ and $y = \frac{b}{a}x$ for a horizontal transverse axis
    - $y = -\frac{a}{b}x$ and $y = \frac{a}{b}x$ for a vertical transverse axis

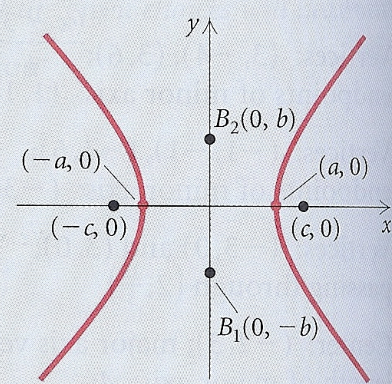
Standard Equation of a Hyperbola with Center at the Origin

Transverse Axis Horizontal

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Vertices: $(-a, 0), (a, 0)$

Foci: $(-c, 0), (c, 0)$, where $c^2 = a^2 + b^2$

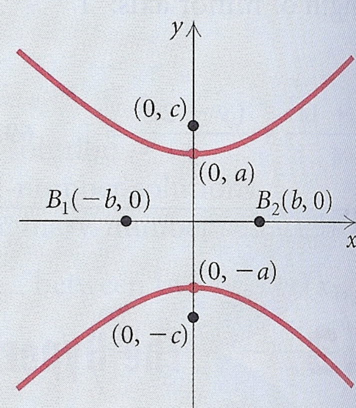


Transverse Axis Vertical

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Vertices: $(0, -a), (0, a)$

Foci: $(0, -c), (0, c)$, where $c^2 = a^2 + b^2$



Example 7. Find the center, the vertices, the foci, and the asymptotes of

$$\frac{x^2}{25} - \frac{y^2}{9} = 1$$

Sketch the graph.

Example 8. Find the center, the vertices, the foci, and the asymptotes of

$$y^2 - 4x^2 = 16.$$

Sketch the graph.

Example 9. Find an equation of a hyperbola satisfying the given conditions:

Vertices at $(1, 0)$ and $(-1, 0)$
foci at $(2, 0)$ and $(-2, 0)$

Homework: pp 542-543; #1–17 odd, #23 –35 odd;
pp 553; #7, 9, 11, 17–23 odd