MATH 11010: Inverse Functions
Section 4.1

• **Inverse Relation:** Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse relation. If the relation is defined by an equation, interchanging the variables produces an equation of the inverse relation.

• **One-to-one functions:** A function \( f \) is one-to-one if no two elements of the domain \( A \) have the same image. In other words, \( f \) is a one-to-one function if \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \).

Example 1: Determine whether \( f(x) = 8x - 3 \) is a one-to-one function.

• **Horizontal Line Test:** If it is possible for a horizontal line to intersect the graph of a function more than once, then the function is NOT one-to-one.

Example 2: Determine if each of the following curves is the graph of an one-to-one function.
• **Inverse function:** Let $f$ be a one-to-one function with domain $A$ and range $B$. Then its inverse function, denoted $f^{-1}$, has domain $B$ and range $A$ and is defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

for any $y$ in $B$. Please note that $^{-1}$ is NOT an exponent; therefore, $f^{-1}$ does NOT mean the reciprocal of $f$.

• **Properties of inverse functions:**

* Let $f$ be a one-to-one function with domain $A$ and range $B$. The inverse function $f^{-1}$ satisfies

$$f^{-1}(f(x)) = x \quad \text{for every} \ x \text{ in } A$$

and

$$f(f^{-1}(x)) = x \quad \text{for every} \ x \text{ in } B$$

* The inverse of $f^{-1}$ is $f$. So, we say that $f$ and $f^{-1}$ are inverses of each other.

* The inverse function interchanges the domain and range. Namely,

$$\text{Domain of } f = \text{Range of } f^{-1}$$

$$\text{Range of } f = \text{Domain of } f^{-1}$$

* The graph of $f^{-1}$ is found by reflecting the graph of $f$ across the line $y = x$.

* Only a one-to-one function can have an inverse function.

**Example 3:** For the function $f$, use composition of functions to show that $f^{-1}$ is as given.

$$f(x) = \frac{x + 5}{4} \quad \text{and} \quad f^{-1}(x) = 4x - 5.$$
Finding the inverse of a one-to-one function:

- Replace \( f(x) \) with \( y \).
- Interchange \( x \) and \( y \).
- Solve this equation for \( y \). The resulting equation is \( f^{-1}(x) \).

**Example 4:** Find the inverse of \( f(x) = 9 - 7x \).

**Example 5:** Find the inverse of \( f(x) = \frac{x + 1}{3x + 2} \).
Example 6: Find the inverse of \( f(x) = 3x^2 - 4, \ x \leq 0. \)

Example 7: Given the graph of \( f \), sketch the graph of \( f^{-1} \).