Section 5.3: Definite Integrals and Areas

Definition. If F is an antiderivative of f, then the **definite integral** of f from a to b is defined as

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a).$$

Here, a is called the **lower limit of integration** and b is called the **upper limit of integration**.

Note. The *indefinite integral* $\int f(x)\,dx$ is the antiderivative plus a constant. That is,

$$\int f(x)\,dx = F(x) + C.$$

The *definite integral* $\int_a^b f(x)\,dx$ is the antiderivative evaluated at $x = b$ minus the antiderivative evaluated at $x = a$. That is,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Hence, when finding a definite integral, your answer will always be a number (positive, negative, or zero). The meaning of this number will be explored in the next Topic.

Example 1. Find $\displaystyle\int_1^3 (2x-1)\,dx$.

SOLUTION.

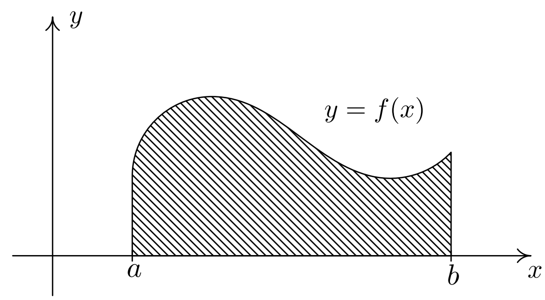
$$\begin{aligned}
\int_1^3 (2x-1)\,dx &= \left(x^2 - x\right)\Big|_{x=1}^{x=3} \\
&= \left[(3)^2 - (3)\right] - \left[(1)^2 - (1)\right] \\
&= [9-3] - [1-1] \\
&= [6] - [0] \\
&= \boxed{6}
\end{aligned}$$

Example 2. Find $\displaystyle\int_1^4 (x^2 - 4x + 3)\,dx.$

Example 3. Find $\displaystyle\int_0^4 \left(\sqrt{x} - 2x - 3\right)\,dx.$

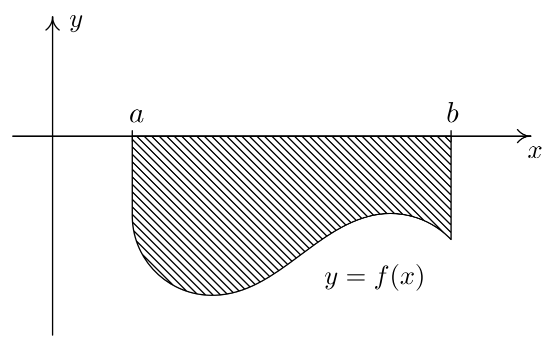
Results:

- If the graph of $y = f(x)$ is *above* the x-axis for $a \le x \le b$, then



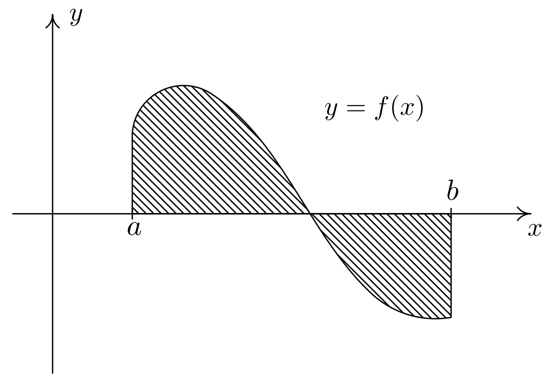
$$\boxed{\int_a^b f(x)\,dx = \text{the area under the curve from } a \text{ to } b}$$

- If the graph of $y = f(x)$ is *below* the x-axis for $a \le x \le b$, then $\int_a^b f(x)\,dx$ will be a *negative number* and



$$\boxed{\int_a^b f(x)\,dx = -(\text{the area above the curve from } a \text{ to } b)}$$

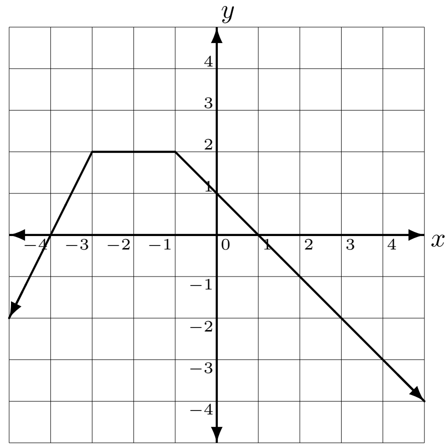
- If the graph of $y = f(x)$ is *above and below* the x-axis for $a \le x \le b$, then



$$\int_a^b f(x)\,dx = \text{the "net" area under/above the curve from } a \text{ to } b$$

Example 4. Find the area under $f(x) = 9 - 3\sqrt{x}$ from $x = 0$ to $x = 9$.

Example 5. For the function f graphed below, find the following using basic area formulas from geometry:



(a) $\int_{-3}^{-1} f(x)\,dx =$

(b) $\int_{1}^{3} f(x)\,dx =$

(c) $\int_{-1}^{2} f(x)\,dx =$

(d) $\int_{-1}^{3} f(x)\,dx =$

EXERCISES

Find the following definite integrals.

1. $\int_{1}^{2} (3x^2 + 4x - 1)\, dx =$

2. $\int_{-1}^{1} (6x^2 - 8x + 2)\, dx =$

3. $\int_{-4}^{-1} (-6x^2 + 2x + 1)\, dx =$

4. $\int_{-2}^{0} (-3x^2 - 4x - 2)\, dx =$

5. $\int_{-2}^{2} (x^2 + 3x + 1)\, dx =$

6. $\int_{-3}^{1} (2x^2 + 4x + 5)\, dx =$

7. $\int_{0}^{1} (4x^3 + x^2 - 5x + 2)\, dx =$

8. $\int_{-1}^{0} (x^4 + x^2 + 1)\, dx =$

9. $\int_{-2}^{1} (-x^2 + x - 5)\, dx =$

10. $\int_{0}^{4} (2\sqrt{x} + x + 1)\, dx =$

11. $\displaystyle\int_1^9 (\sqrt{x} + 3x - 2)\,dx =$

12. $\displaystyle\int_0^1 (\sqrt{x} - x^3)\,dx =$

13. $\displaystyle\int_1^e \frac{2}{x}\,dx =$

14. $\displaystyle\int_{-3}^{-1} \frac{4}{x}\,dx =$

15. $\displaystyle\int_0^2 e^{2x}\,dx =$

16. $\displaystyle\int_0^1 e^{-x}\,dx =$

ANSWERS

1. 12
2. 8
3. -138
4. -4
5. $28/3$
6. $68/3$
7. $5/6$
8. $23/15$
9. $-39/2$
10. $68/3$
11. $364/3$
12. $5/12$
13. 2
14. $-4\ln 3$
15. $\frac{1}{2}e^4 - \frac{1}{2}$
16. $-e^{-1} + 1$