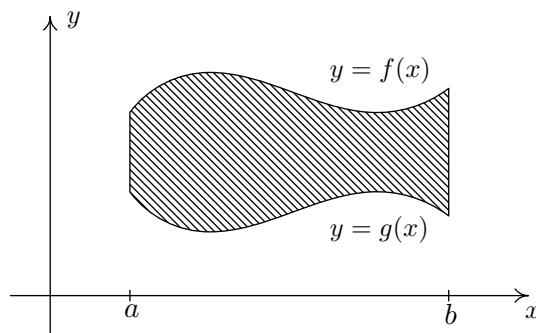# Section 5.4: Average Value and Area Between Curves

Definition. The **average value of a continuous function** on an interval $[a, b]$ is given by:

$$\text{average value on } [a, b] = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

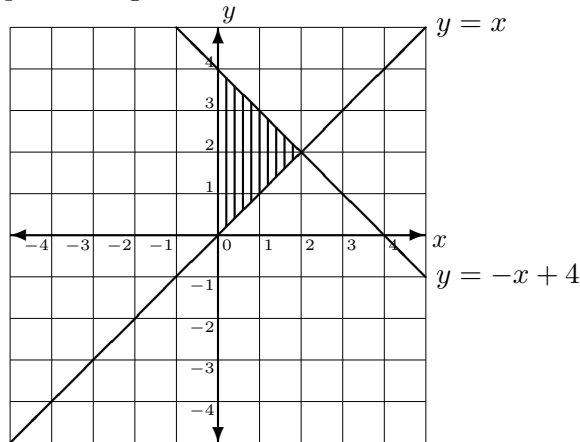**Example 1.** Find the average value of $f(x) = 16 - 3x^2$ on $[-2, 3]$.

Area Between Curves Result. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for $a \leq x \leq b$ is

$$\text{Area} = \int_a^b \big[f(x) - g(x) \big]\, dx = \int_a^b \big[\text{Top function} - \text{Bottom function} \big]\, dx$$

Example 2. Find the area of the region bounded by the graphs of $y = -x + 4$, $y = x$, and $x = 0$.
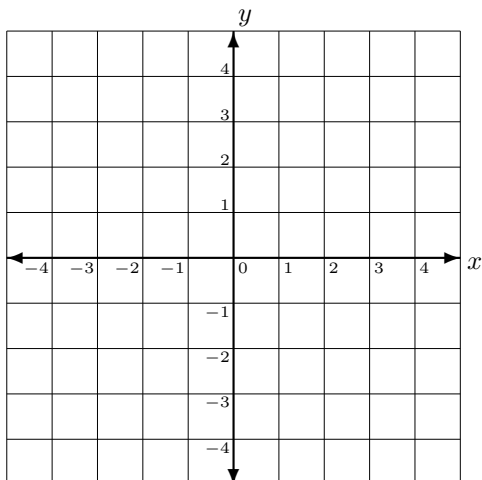
SOLUTION. We must first graph this region:

Then

$$\text{Area} = \int_0^2 \left[\text{Top function} - \text{Bottom function}\right] dx = \int_0^2 \left[(-x + 4) - (x)\right] dx.$$

$$\text{Area} = \int_0^2 \left[(-x + 4) - (x)\right] dx$$

$$= \int_0^2 \left[-2x + 4\right] dx$$

$$= -x^2 + 4x \Big|_{x=0}^{x=2}$$

$$= \left[-(2)^2 + 4(2)\right] - \left[-(0)^2 + 4(0)\right]$$

$$= \left[-(4) + 8\right] - \left[-0 + 0\right]$$

$$= \boxed{4}$$

Important Note. The problems in this section require you to find the area of a region. Since a region can never have negative area, your final answer must be a positive number. (Note that a definite integral can be negative-valued, or equal to zero, but the area of a region must always be positive-valued.) Therefore, if your final answer happens to be negative, then you made an error somewhere in your solution.
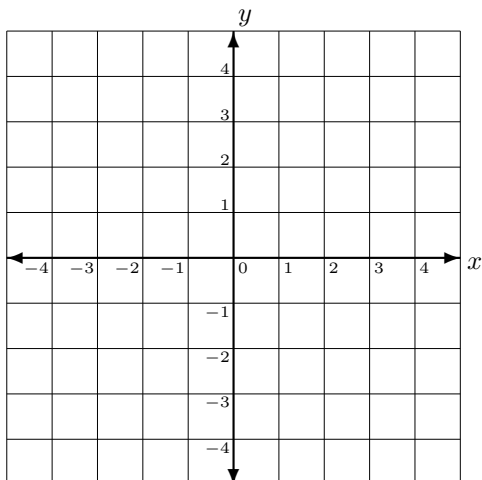
Example 3. Find the area of the region bounded by the graphs of $y = x^2$, $x = 2$, and the x-axis. Be sure to first accurately graph this region on the axes below.

$\boxed{Answer:\ 8/3}$
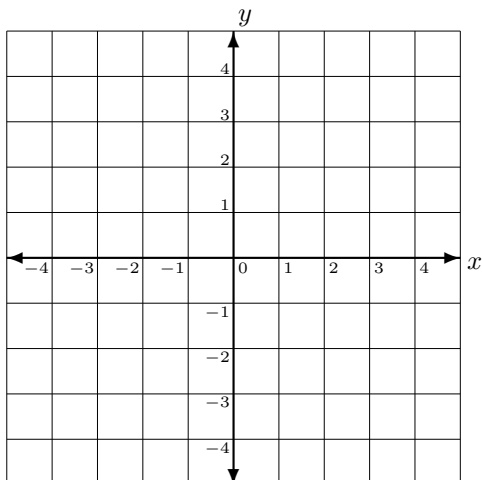
Example 4. Find the area of the region bounded by the graphs of $\quad y = \sqrt{x}, \quad x = 1, \quad x = 4,$ and the x-axis. Be sure to first accurately graph this region on the axes below.
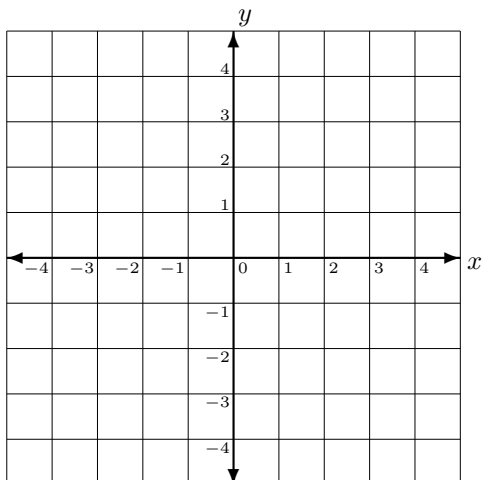
$\boxed{Answer:\ 14/3}$

Example 5. Find the area of the region bounded by the graphs of $y = \sqrt{x} + 2$, $y = x$, and $x = 0$. Be sure to first accurately graph this region on the axes below.
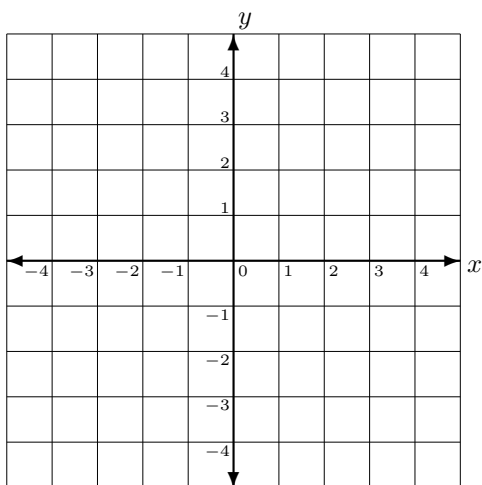
$\boxed{Answer:\ 16/3}$

Example 6. Find the area of the region bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$. Be sure to first accurately graph this region on the axes below.
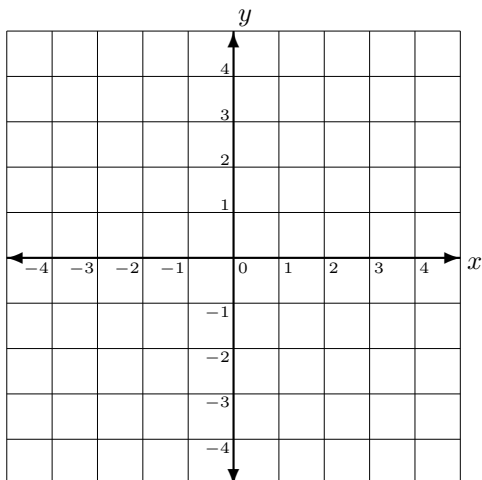
$\boxed{Answer:\ 1/3}$

Example 7. Find the area of the region bounded by the graphs of $y = \sqrt{x}$ and $y = \dfrac{1}{2}x$.
Be sure to first accurately graph this region on the axes below.

$\boxed{Answer:\ 4/3}$

Example 8. Find the area of the region bounded by the graphs of $y = x^2 - 2$ and $y = x$. Be sure to first accurately graph this region on the axes below.
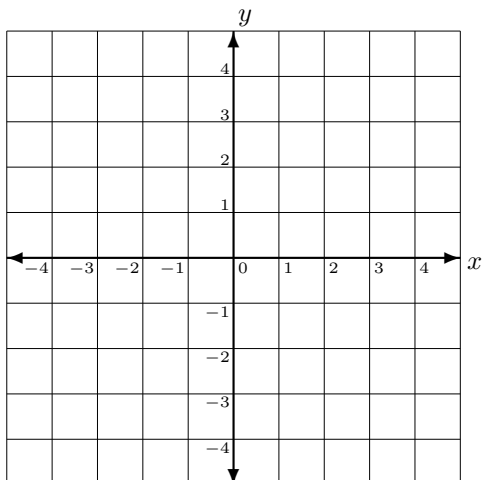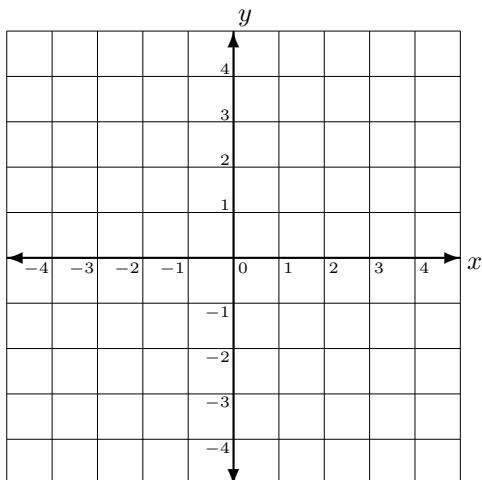
$Answer$: $9/2$

Example 9. Find the area of the region bounded by the graphs of $y = -x^2 + 3$ and $y = -x + 1$. Be sure to first accurately graph this region on the axes below.

$\boxed{\textit{Answer: } 9/2}$

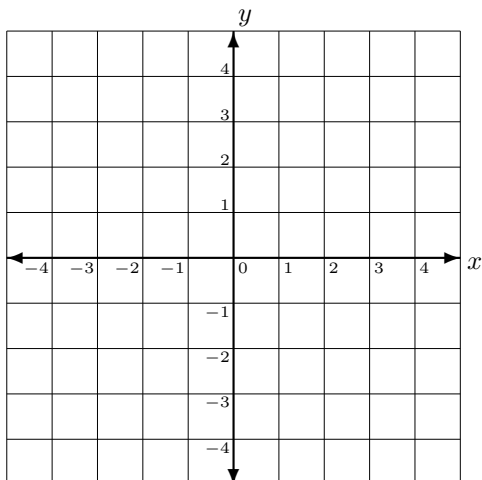**Example 10.** Find the area of the region bounded by the graphs of $y = x^2$, $y = 4$, and $x \geq 0$. Be sure to first accurately graph this region on the axes below.
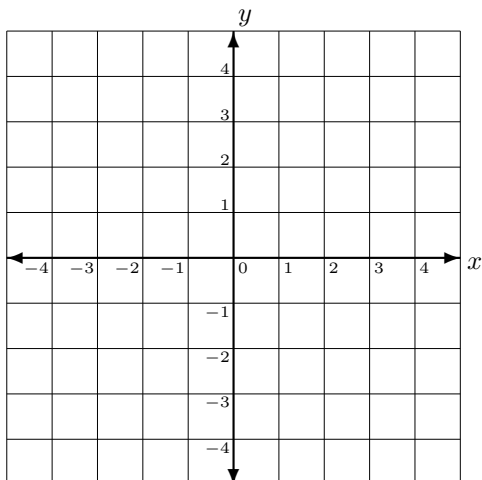
$Answer$: $16/3$

Example 11. Find the area of the region bounded by the graphs of $y = \sqrt{x}$, $y = 2$, and $x = 0$. Be sure to first accurately graph this region on the axes below.

$\boxed{\textit{Answer: } 8/3}$

Example 12. Find the area of the region bounded by the graphs of $y = x^3$ and $y = x$. Be sure to first accurately graph this region on the axes below.
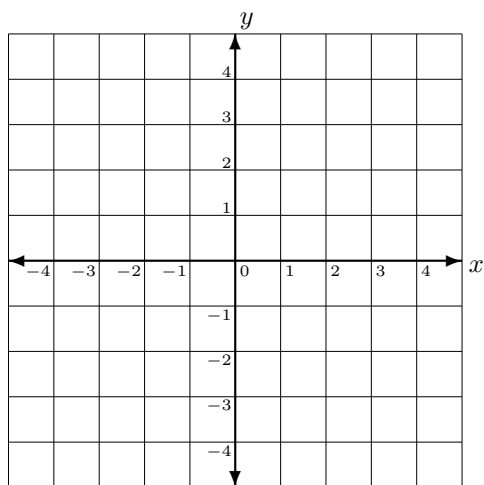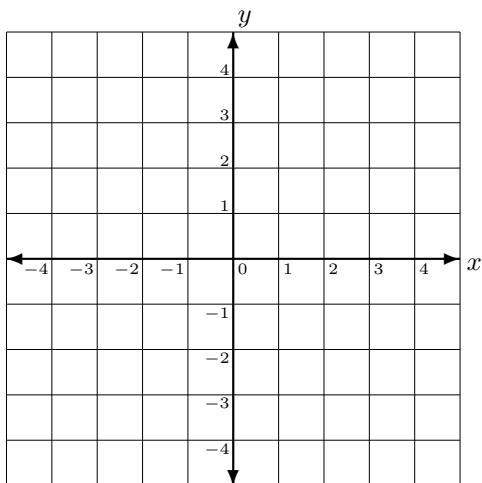
$\boxed{Answer:\ 1/2}$

EXERCISES

1. Find the area of the region bounded by the graphs of $y = x^3 + 3$, $y = -2x$, and $x = 1$. Be sure to first accurately graph this region on the axes below.
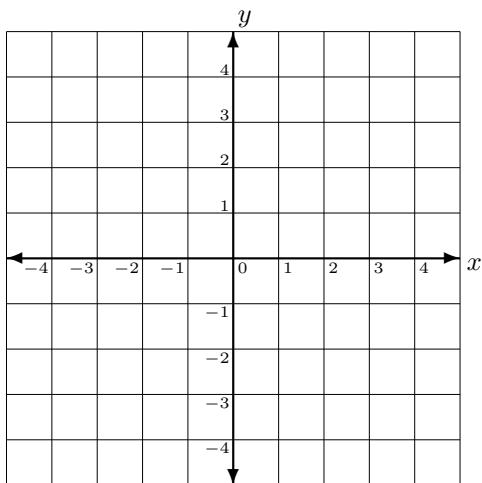
$$\boxed{Answer:\ 6}$$

2. Find the area of the region bounded by the graphs of $y = -\sqrt{x} + 3,$ $y = 1,$ and $x = 0.$ Be sure to first accurately graph this region on the axes below.

$\boxed{Answer: 8/3}$

3. Find the area of the region bounded by the graphs of $\;y = \dfrac{1}{4}x + 1,\quad y = -\dfrac{1}{2}x + 4,$
   and $\;x = 0.\;$ Be sure to first accurately graph this region on the axes below.

$\boxed{Answer:\ 6}$

4. Find the area of the region bounded by the graphs of $y = x^3 + 2$, $y = -\sqrt{x} + 2$, and $x = 1$. Be sure to first accurately graph this region on the axes below.

$\boxed{\textit{Answer}: 11/12}$