Section 2.2: Derivatives

Definition. For a function f, its **derivative** f' is another function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists. If the derivative exists at $x = a$, we say f *is differentiable at* a.

THE BIG RESULT

Graphically, $f'(a)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point $\big(a, f(a)\big)$. That is,

$$m_{\text{tan at } x = a} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Since the derivative is a slope, we have the following immediate results (see page **??**):

- If $f'(a)$ is positive, then the slope of the tangent line is positive and hence the graph of the function is *increasing* ($\nearrow$) at $x = a$.

- If $f'(a)$ is negative, then the slope of the tangent line is negative and hence the graph of the function is *decreasing* ($\searrow$) at $x = a$.

- If $f'(a) = 0$, then the slope of the tangent line is zero and hence the graph has a *horizontal tangent* ($\rightarrow$) at $x = a$.

- If $f'(a)$ is a large positive number, then the slope of the tangent line is a large positive number and hence the tangent line is near vertical. In this case, we say the function is *rapidly increasing* at $x = a$.

- If $f'(a)$ is a small positive number, then the slope of the tangent line is a small positive number and hence the tangent line is near horizontal. In this case, we say the function is *slowly increasing* at $x = a$.

Notation. Let $y = f(x)$.

- The derivative may be denoted in the following ways:

$$f'(x) \qquad \text{OR} \qquad \frac{d}{dx}f(x) \qquad \text{OR} \qquad y' \qquad \text{OR} \qquad \frac{dy}{dx}.$$

- The derivative evaluated at a value of x, say for example $x = 5$, may be denoted in the following ways:

$$f'(5) \qquad \text{OR} \qquad \frac{d}{dx}f(x)\Big|_{x=5} \qquad \text{OR} \qquad y'\Big|_{x=5} \qquad \text{OR} \qquad \frac{dy}{dx}\Big|_{x=5}.$$

Example 1. If $f(x) = x^2 - 4x + 1$, find f' using the definition.

SOLUTION.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left[(x+h)^2 - 4(x+h) + 1\right] - \left[x^2 - 4x + 1\right]}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{4x} - 4h + \cancel{1} - \cancel{x^2} + \cancel{4x} - \cancel{1}}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 4h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 4)}{\cancel{h}} \\
&= \lim_{h\to 0} 2x + h - 4 \\
&= \boxed{2x - 4}
\end{aligned}$$

Example 2. Using the result in Example 1,

(a) find the slope of the tangent line to the graph of $f(x) = x^2 - 4x + 1$ at the point $(3, -2)$.

(b) find the slope of the tangent line to the graph of $f(x) = x^2 - 4x + 1$ at the point $(2, -3)$.

SOLUTION.

(a) The slope of the tangent line is given by the derivative. Hence, we need to evaluate the derivative at $x = 3$, the x-coordinate of the point $(3, -2)$. (Note that the derivative is $f'(x) = 2x - 4$, as calculated in Example 1 above.)

$$m_{\text{tan at } x=3} = f'(3) = 2(3) - 4 = 6 - 4 = \boxed{2}.$$

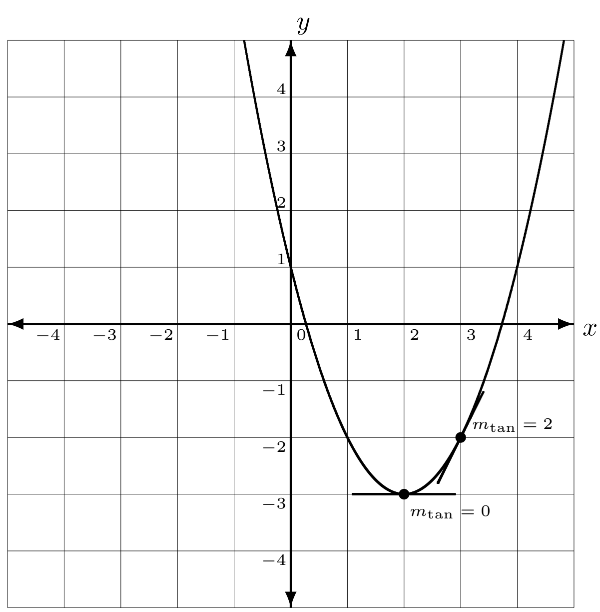
Since the derivative is positive, the slope of the tangent line is positive, and therefore the graph of the function is *increasing* ($\nearrow$) at $x = 3$.

(b) The solution is identical to part (a), except we now evaluate the derivative at $x = 2$, the x-coordinate of the point $(2, -3)$.

$$m_{\text{tan at } x=2} = f'(2) = 2(2) - 4 = 4 - 4 = \boxed{0}.$$

Since the derivative is zero, the slope of the tangent line is zero, and therefore the graph of the function has a *horizontal tangent* ($\rightarrow$) at $x = 2$.

Here are the results of Example 2 illustrated on the graph of $f(x) = x^2 - 4x + 1$:



Example 3. Let $f(x) = x^3 - 3x$.

(a) Find $f'(x)$ using the definition.

(b) Find the slope of the tangent line to the graph of $y = f(x)$ at the point $(2, 2)$.

(c) Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(2, 2)$.

(d) Find the value(s) of x at which the graph of $y = f(x)$ has a horizontal tangent.

Example 4. Find the equation of the tangent line to $f(x) = 5 + 3x - 4x^2$ at $x = 1$.

Supplemental Exercises

Find the derivative of the following functions using the definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

1. $f(x) = x^2 + 5x - 1$

2. $f(x) = 2x^2 + 3x + 1$

3. $f(x) = x^3 + 5$

4. $f(x) = \frac{1}{x}$

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5. If $f(x) = x^2 - 4x + 2$, then it will soon be shown that its derivative is

$$f'(x) = 2x - 4.$$

   (a) Find the slope of the tangent line to the graph of $y = f(x)$ at the point $(1, -1)$.
   (b) Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(1, -1)$.
   (c) Find the value(s) of x at which the graph of $y = f(x)$ has a horizontal tangent.
   (d) Find the value(s) of x at which the graph of $y = f(x)$ has a tangent line having slope 4.

6. If $f(x) = \frac{4}{3}x^3 - x - 26$, then it will soon be shown that its derivative is

$$f'(x) = 4x^2 - 1.$$

   (a) Find the slope of the tangent line to the graph of $y = f(x)$ at the point $(3, 7)$.
   (b) Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(3, 7)$.
   (c) Find the value(s) of x at which the graph of $y = f(x)$ has a horizontal tangent.
   (d) Find the value(s) of x at which the graph of $y = f(x)$ has a tangent line having slope 3.
   (e) Find the value(s) of x at which the graph of $y = f(x)$ has a tangent line having slope 7.

7. If $f(x) = x^3 + x^2 - x - 2$, then it will soon be shown that its derivative is

$$f'(x) = 3x^2 + 2x - 1.$$

(a) Find the slope of the tangent line to the graph of $y = f(x)$ at the point $(1, -1)$.

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(1, -1)$.

(c) Find the value(s) of x at which the graph of $y = f(x)$ has a horizontal tangent.

(d) Find the value(s) of x at which the graph of $y = f(x)$ has a tangent line having slope 1.

8. If $f(x) = \dfrac{x}{x^2+1}$, then it will soon be shown that its derivative is

$$f'(x) = \frac{1 - x^2}{(x^2+1)^2}.$$

(a) Find the slope of the tangent line to the graph of $y = f(x)$ at the point $(2, 2/5)$.

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(2, 2/5)$.

(c) Find the value(s) of x at which the graph of $y = f(x)$ has a horizontal tangent.

9. If $f(x) = \frac{2}{3}(x+2)^{3/2}$, then it will soon be shown that its derivative is

$$f'(x) = \sqrt{x+2}.$$

(a) Find the slope of the tangent line to the graph of $y = f(x)$ at the point $(7, 18)$.

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(7, 18)$.

(c) Find the value(s) of x at which the graph of $y = f(x)$ has a horizontal tangent.

(d) Find the value(s) of x at which the graph of $y = f(x)$ has a tangent line having slope 4.

10. If $f(x) = 3xe^x$, then it will soon be shown that its derivative is

$$f'(x) = 3e^x + 3xe^x.$$

(a) Find the slope of the tangent line to the graph of $y = f(x)$ at the point $(0, 0)$.

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(0, 0)$.

(c) Find the value(s) of x at which the graph of $y = f(x)$ has a horizontal tangent.

11. If $f(x) = \dfrac{\ln x}{2x}$, then it will soon be shown that its derivative is

$$f'(x) = \frac{1 - \ln x}{2x^2}.$$

(a) Find the slope of the tangent line to the graph of $y = f(x)$ at the point $(1, 0)$.

(b) Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(1, 0)$.

(c) Find the value(s) of x at which the graph of $y = f(x)$ has a horizontal tangent.

ANSWERS

1. $f'(x) = 2x + 5$

2. $f'(x) = 4x + 3$

3. $f'(x) = 3x^2$

4. $f'(x) = -\dfrac{1}{x^2}$

5. (a) -2
   (b) $y = -2x + 1$
   (c) $x = 2$
   (d) $x = 4$

6. (a) 35
   (b) $y = 35x - 98$
   (c) $x = \pm 1/2$
   (d) $x = \pm 1$
   (e) $x = \pm\sqrt{2}$

7. (a) 4
   (b) $y = 4x - 5$
   (c) $x = 1/3, \quad -1$
   (d) $x = -\frac{1}{3} \pm \frac{\sqrt{7}}{3}$

8. (a) $-3/25$
   (b) $y = -\frac{3}{25}x + \frac{16}{25}$
   (c) $x = \pm 1$

9. (a) 3
   (b) $y = 3x - 3$
   (c) $x = -2$
   (d) $x = 14$

10. (a) 3
    (b) $y = 3x$
    (c) $x = -1$

11. (a) $1/2$
    (b) $y = \frac{1}{2}x - \frac{1}{2}$
    (c) $x = e$